Demo 1

This demo exemplifies the techniques of the three clipping algorithms on a polynomial of 5th degree. The example was used in preparation of the associated talk, because most of the different cases occurring during the algorithms' execution can easily be found herein.
This demo works with the standard datatype `double` and precision $\varepsilon = 0.001$.

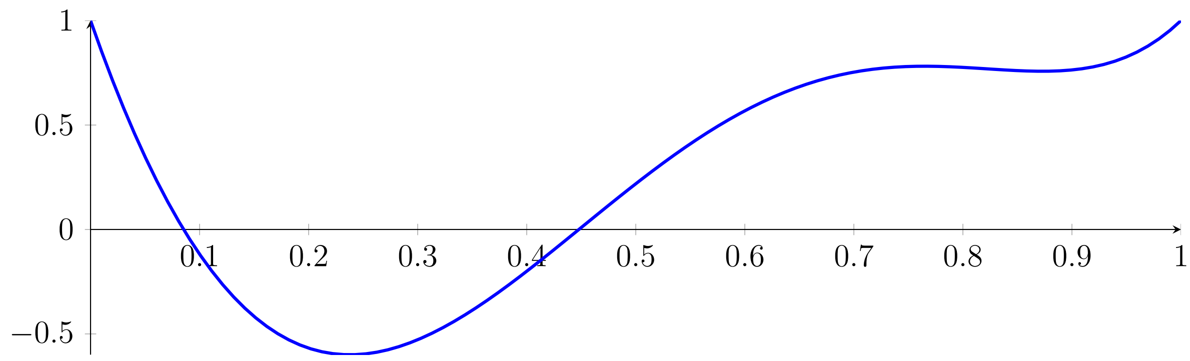
Contents

1 `BezClip` Applied to a Polynomial of 5th Degree with Two Roots

$$25X^5 - 35X^4 - 15X^3 + 40X^2 - 15X + 1$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

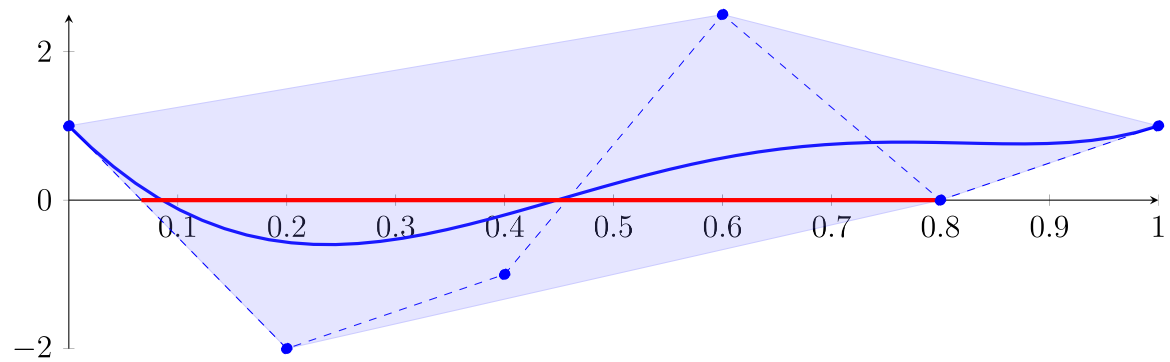
$$p = 25X^5 - 35X^4 - 15X^3 + 40X^2 - 15X + 1$$



1.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 25X^5 - 35X^4 - 15X^3 + 40X^2 - 15X + 1 \\ &= 1B_{0,5}(X) - 2B_{1,5}(X) - 1B_{2,5}(X) + 2.5B_{3,5}(X) + 0B_{4,5}(X) + 1B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0666667, 0.8\}$$

Intersection intervals with the x axis:

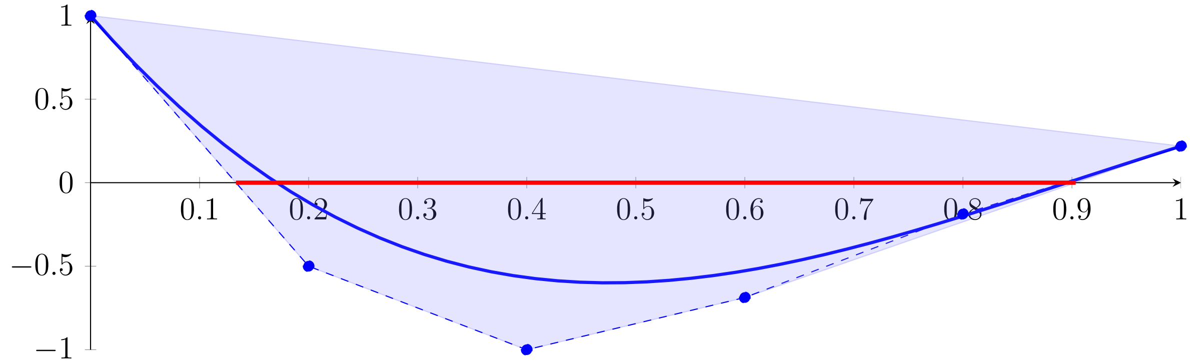
$$[0.0666667, 0.8]$$

Longest intersection interval: 0.733333
$\implies$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

1.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.78125X^5 - 2.1875X^4 - 1.875X^3 + 10X^2 - 7.5X + 1 \\ &= 1B_{0,5}(X) - 0.5B_{1,5}(X) - 1B_{2,5}(X) - 0.6875B_{3,5}(X) - 0.1875B_{4,5}(X) + 0.21875B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.133333, 0.903448\}$$

Intersection intervals with the x axis:

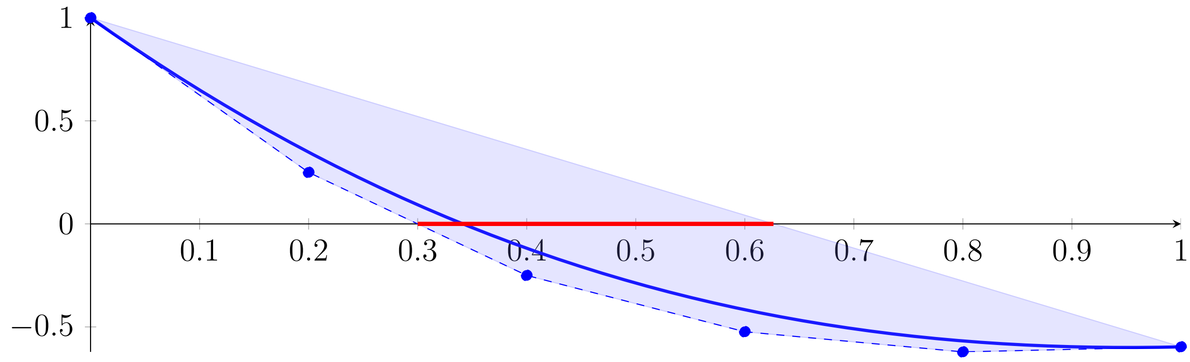
$$[0.133333, 0.903448]$$

Longest intersection interval: 0.770115
$\Longrightarrow$ Bisection: first half $[0, 0.25]$ und second half $[0.25, 0.5]$

1.3 Recursion Branch 1 1 1 on the First Half $[0, 0.25]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 0.0244141X^5 - 0.136719X^4 - 0.234375X^3 + 2.5X^2 - 3.75X + 1$$
$$= 1B_{0,5}(X) + 0.25B_{1,5}(X) - 0.25B_{2,5}(X) - 0.523438B_{3,5}(X) - 0.621094B_{4,5}(X) - 0.59668B_{5,5}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.6263\}$$

Intersection intervals with the x axis:

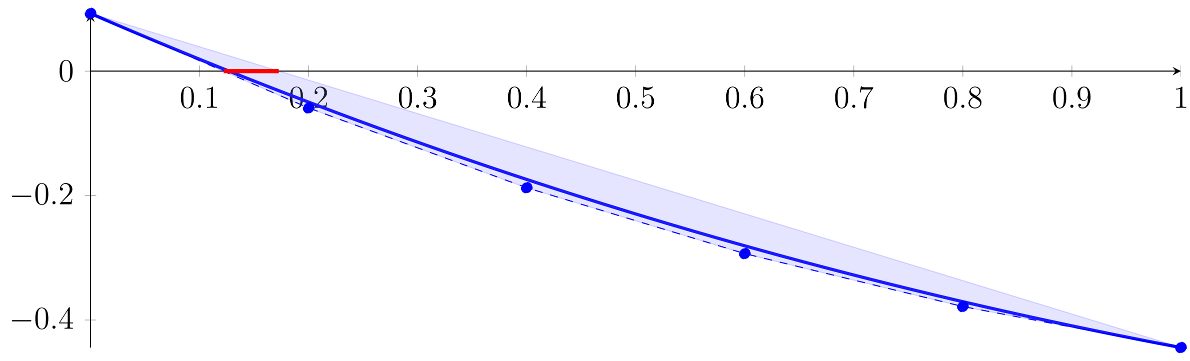
$$[0.3, 0.6263]$$

Longest intersection interval: 0.3263
$\Longrightarrow$ Selective recursion: interval 1: $[0.075, 0.156575]$,

1.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.075, 0.156575]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 9.03074 \cdot 10^{-05} X^5 - 0.00113472 X^4 - 0.013079 X^3 + 0.236561 X^2 - 0.759318 X + 0.0926238$$
$$= 0.0926238 B_{0,5}(X) - 0.0592399 B_{1,5}(X) - 0.187447 B_{2,5}(X)$$
$$- 0.293307 B_{3,5}(X) - 0.378353 B_{4,5}(X) - 0.444257 B_{5,5}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.121983, 0.172522\}$$

Intersection intervals with the x axis:

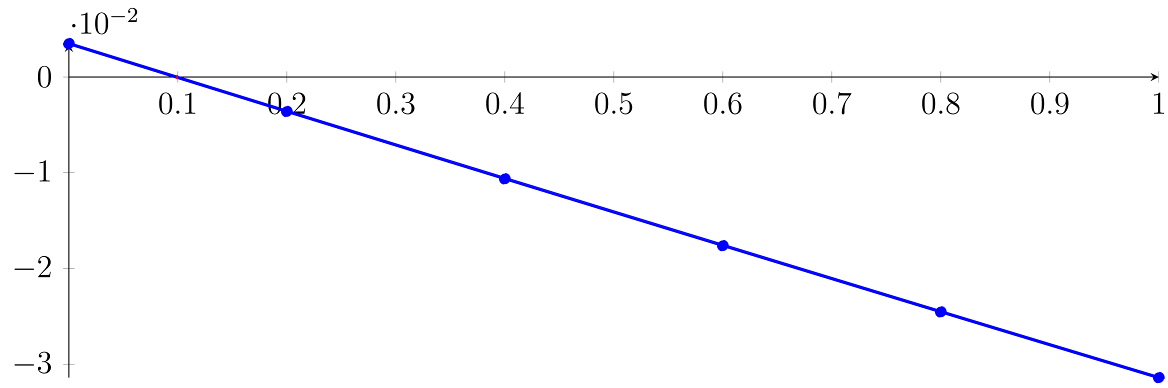
$$[0.121983, 0.172522]$$

Longest intersection interval: 0.0505393
$\Longrightarrow$ Selective recursion: interval 1: $[0.0849507, 0.0890735]$,

1.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.0849507, 0.0890735]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 2.97762 \cdot 10^{-11} X^5 - 7.04366 \cdot 10^{-09} X^4 - 1.75809 \cdot 10^{-06} X^3 + 0.00059175 X^2 - 0.0354886 X + 0.003496$$
$$= 0.003496 B_{0,5}(X) - 0.00360172 B_{1,5}(X) - 0.0106403 B_{2,5}(X)$$
$$- 0.0176198 B_{3,5}(X) - 0.0245405 B_{4,5}(X) - 0.0314026 B_{5,5}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0985104, 0.100176\}$$

Intersection intervals with the x axis:

$$[0.0985104, 0.100176]$$

Longest intersection interval: 0.00166539
$\Longrightarrow$ Selective recursion: interval 1: $[0.0853569, 0.0853637]$,

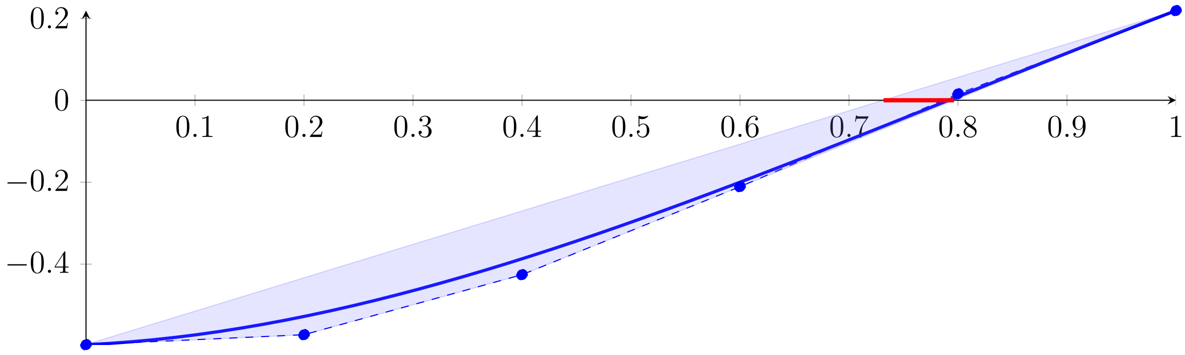
1.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.0853569, 0.0853637]$

Found root in interval $[0.0853569, 0.0853637]$ at recursion depth 6!

1.7 Recursion Branch 1 1 2 on the Second Half $[0.25, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.0244141X^5 - 0.0146484X^4 - 0.537109X^3 + 1.2207X^2 + 0.12207X - 0.59668 \\ &= -0.59668B_{0,5}(X) - 0.572266B_{1,5}(X) - 0.425781B_{2,5}(X) \\ &\quad - 0.210937B_{3,5}(X) + 0.015625B_{4,5}(X) + 0.21875B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.731737, 0.796364\}$$

Intersection intervals with the x axis:

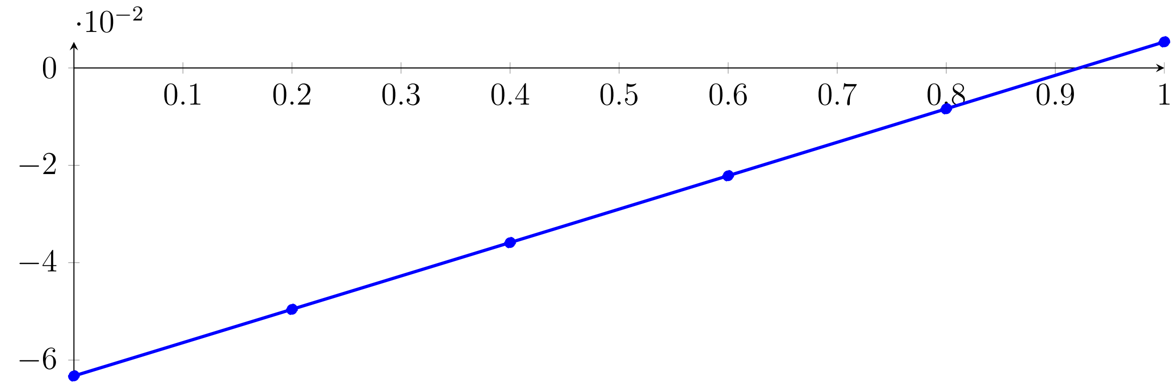
$$[0.731737, 0.796364]$$

Longest intersection interval: 0.0646271
$\Longrightarrow$ Selective recursion: interval 1: $[0.432934, 0.449091]$,

1.8 Recursion Branch 1 1 2 1 in Interval 1: $[0.432934, 0.449091]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 2.75241 \cdot 10^{-08}X^5 + 1.30267 \cdot 10^{-06}X^4 - 0.000121267X^3 + 0.000376859X^2 + 0.0683632X - 0.0632624 \\ &= -0.0632624B_{0,5}(X) - 0.0495898B_{1,5}(X) - 0.0358794B_{2,5}(X) \\ &\quad - 0.0221436B_{3,5}(X) - 0.00839399B_{4,5}(X) + 0.0053577B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.921922, 0.922079\}$$

Intersection intervals with the x axis:

$$[0.921922, 0.922079]$$

Longest intersection interval: 0.000157093
$\Longrightarrow$ Selective recursion: interval 1: $[0.447829, 0.447832]$,

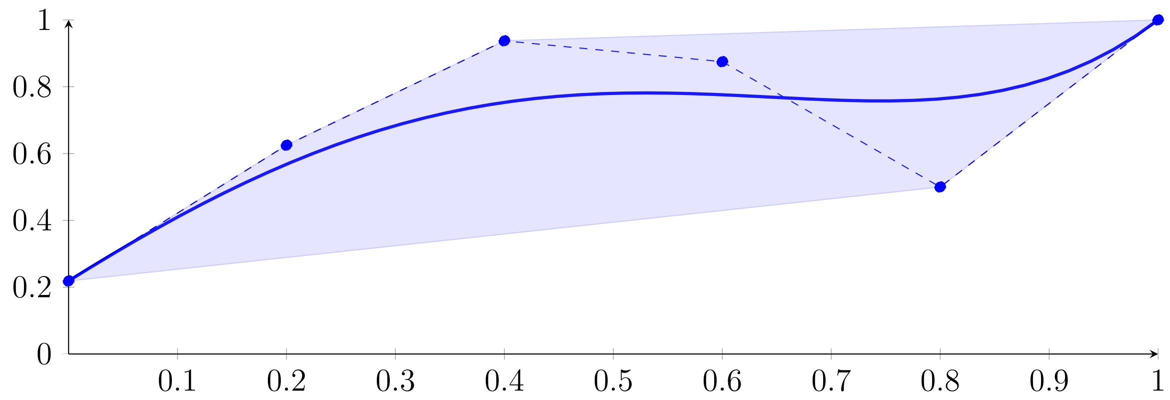
1.9 Recursion Branch 1 1 2 1 1 in Interval 1: $[0.447829, 0.447832]$

Found root in interval $[0.447829, 0.447832]$ at recursion depth 5!

1.10 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.78125X^5 + 1.71875X^4 - 2.8125X^3 - 0.9375X^2 + 2.03125X + 0.21875 \\ &= 0.21875B_{0,5}(X) + 0.625B_{1,5}(X) + 0.9375B_{2,5}(X) + 0.875B_{3,5}(X) + 0.5B_{4,5}(X) + 1B_{5,5}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

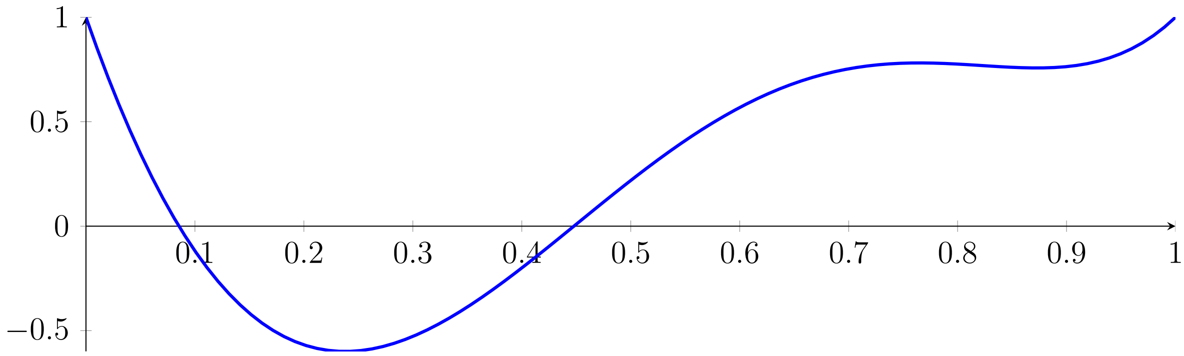
$$\{\}$$

Intersection intervals with the x axis:
No intersection with the x axis. Done.

1.11 Result: 2 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = 25X^5 - 35X^4 - 15X^3 + 40X^2 - 15X + 1$$



Result: Root Intervals

$$[0.0853569, 0.0853637],\ [0.447829, 0.447832]$$

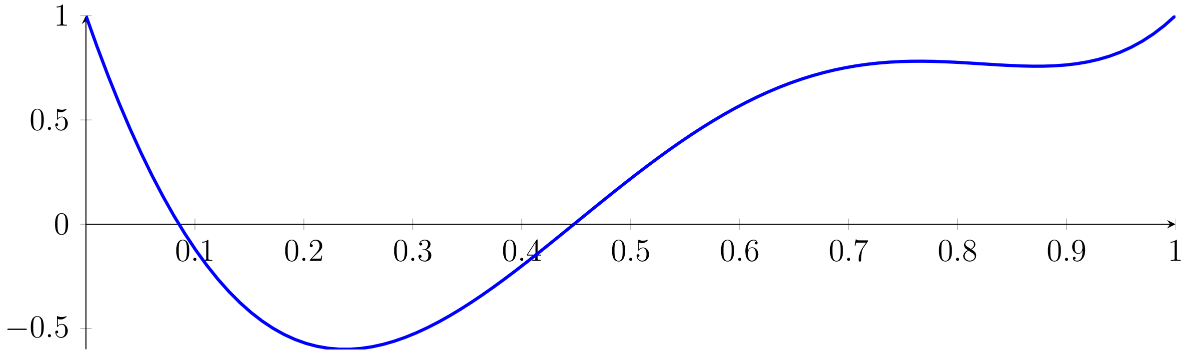
with precision $\varepsilon = 0.001$.

2 QuadClip Applied to a Polynomial of 5th Degree with Two Roots

$$25X^5 - 35X^4 - 15X^3 + 40X^2 - 15X + 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

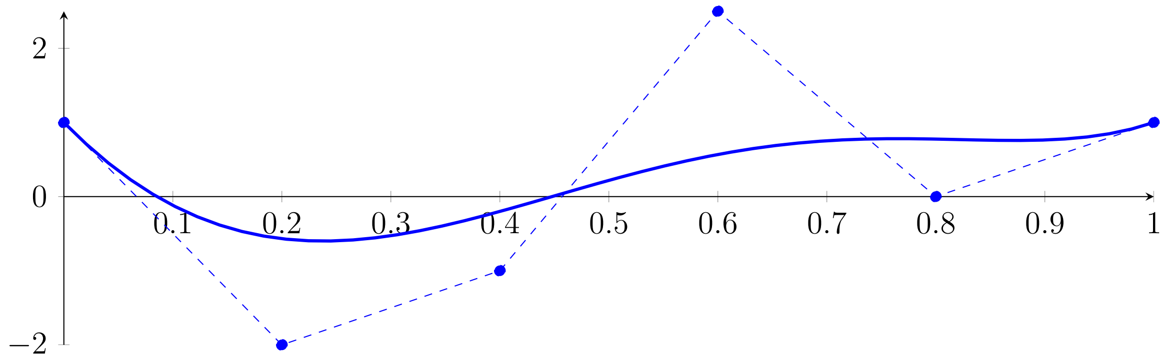
$$p = 25X^5 - 35X^4 - 15X^3 + 40X^2 - 15X + 1$$



2.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 25X^5 - 35X^4 - 15X^3 + 40X^2 - 15X + 1 \\ &= 1B_{0,5}(X) - 2B_{1,5}(X) - 1B_{2,5}(X) + 2.5B_{3,5}(X) + 0B_{4,5}(X) + 1B_{5,5}(X) \end{aligned}$$



Best approximation polynomials of degree 0, 1, 2, 3 and 4:

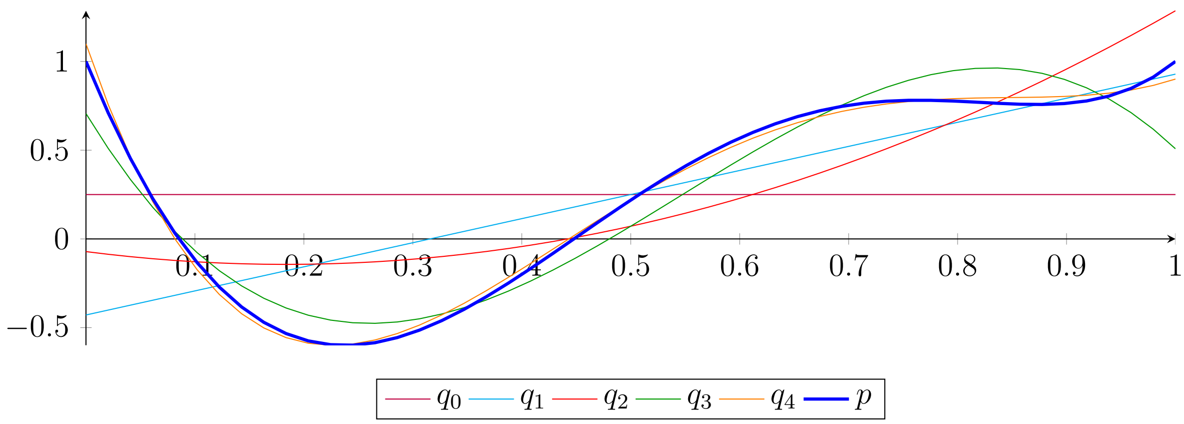
$$\begin{aligned} q_0 &= 0.25 \\ &= 0.25B_{0,0} \end{aligned}$$

$$\begin{aligned} q_1 &= 1.35714X - 0.428571 \\ &= -0.428571B_{0,1} + 0.928571B_{1,1} \end{aligned}$$

$$\begin{aligned} q_2 &= 2.14286X^2 - 0.785714X - 0.0714286 \\ &= -0.0714286B_{0,2} - 0.464286B_{1,2} + 1.28571B_{2,2} \end{aligned}$$

$$\begin{aligned} q_3 &= -15.5556X^3 + 25.4762X^2 - 10.119X + 0.706349 \\ &= 0.706349B_{0,3} - 2.66667B_{1,3} + 2.45238B_{2,3} + 0.507937B_{3,3} \end{aligned}$$

$$\begin{aligned} q_4 &= 27.5X^4 - 70.5556X^3 + 60.8333X^2 - 17.9762X + 1.09921 \\ &= 1.09921B_{0,4} - 3.39484B_{1,4} + 2.25B_{2,4} + 0.394841B_{3,4} + 0.900794B_{4,4} \end{aligned}$$



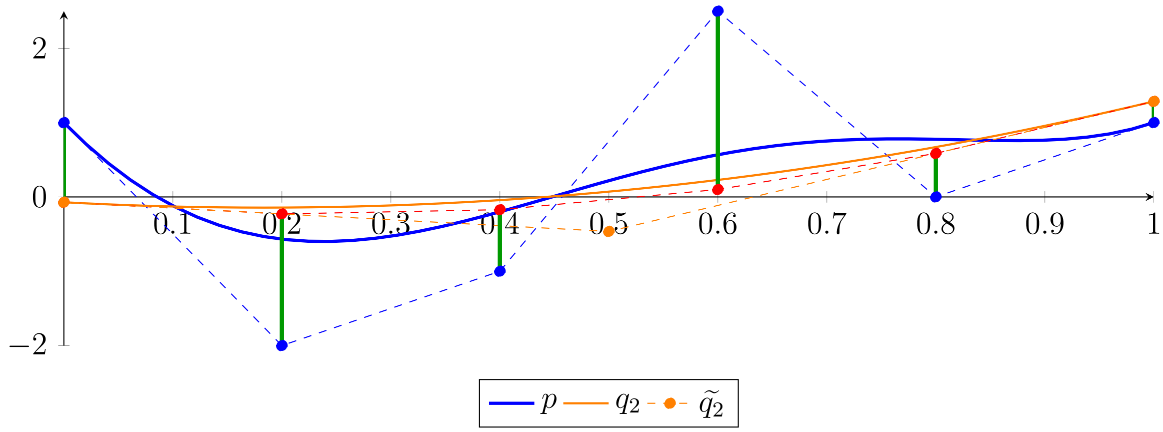
Degree reduction and raising matrices:

$$M_{5,2} = \begin{pmatrix} 0.821429 & -0.428571 & 0.107143 \\ 0.321429 & 0.285714 & -0.107143 \\ 1.249\cdot 10^{-16} & 0.642857 & -0.142857 \\ -0.142857 & 0.642857 & 0 \\ -0.107143 & 0.285714 & 0.321429 \\ 0.107143 & -0.428571 & 0.821429 \end{pmatrix} \quad M_{2,5} = \begin{pmatrix} 1 & 0.6 & 0.3 & 0.1 & 2.1760 \\ 8.10463\cdot 10^{-15} & 0.4 & 0.6 & 0.6 & 0. \\ 4.996\cdot 10^{-15} & -1.59872\cdot 10^{-14} & 0.1 & 0.3 & 0. \end{pmatrix}$$

Degree reduction and raising:

$$\begin{aligned} q_2 &= 2.14286X^2 - 0.785714X - 0.0714286 \\ &= -0.0714286B_{0,2} - 0.464286B_{1,2} + 1.28571B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.18767\cdot 10^{-12}X^5 + 2.52388\cdot 10^{-12}X^4 - 1.79037\cdot 10^{-12}X^3 + 2.14286X^2 - 0.785714X - 0.0714286 \\ &= -0.0714286B_{0,5} - 0.228571B_{1,5} - 0.171429B_{2,5} + 0.1B_{3,5} + 0.585714B_{4,5} + 1.28571B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.4$.
Bounding polynomials M and m:

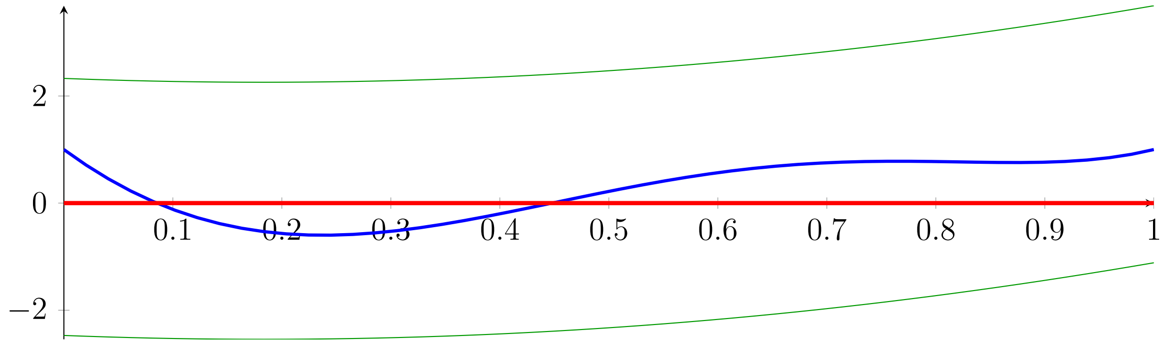
$$M = 2.14286X^2 - 0.785714X + 2.32857$$

$$m = 2.14286X^2 - 0.785714X - 2.47143$$

Root of M and m:

$$N(M) = \{\} \qquad N(m) = \{-0.906136, 1.2728\}$$

Intersection intervals:



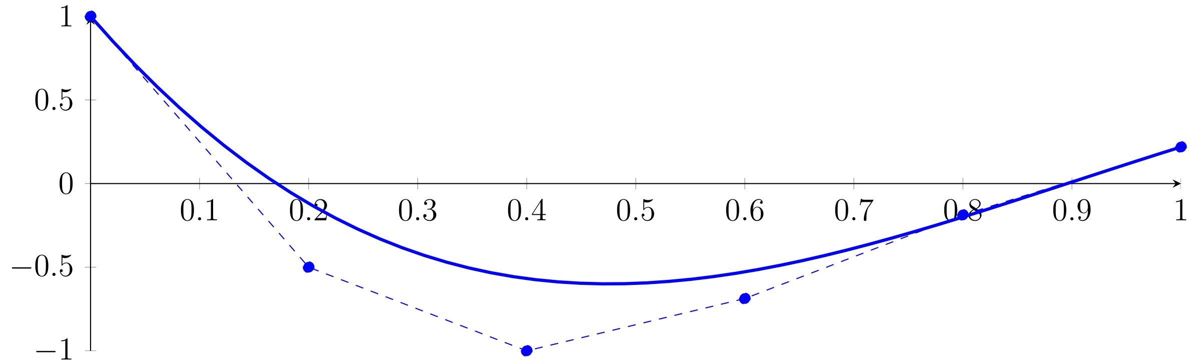
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

2.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 0.78125X^5 - 2.1875X^4 - 1.875X^3 + 10X^2 - 7.5X + 1$$
$$= 1B_{0,5}(X) - 0.5B_{1,5}(X) - 1B_{2,5}(X) - 0.6875B_{3,5}(X) - 0.1875B_{4,5}(X) + 0.21875B_{5,5}(X)$$



Best approximation polynomials of degree 0, 1, 2, 3 and 4:

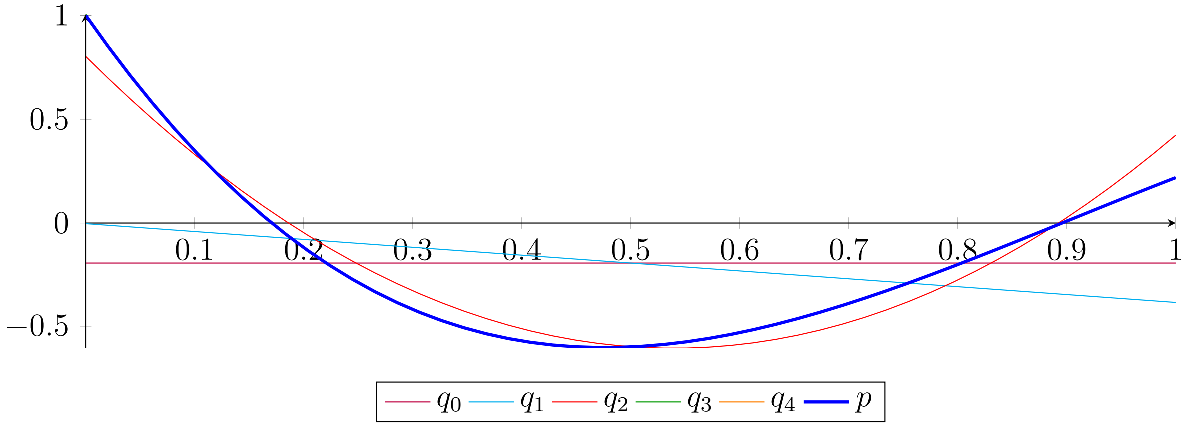
$$q_0 = -0.192708$$
$$= -0.192708B_{0,0}$$

$$q_1 = -0.379464X - 0.00297619$$
$$= -0.00297619B_{0,1} - 0.38244B_{1,1}$$

$$q_2 = 4.83259X^2 - 5.21205X + 0.802455$$
$$= 0.802455B_{0,2} - 1.80357B_{1,2} + 0.422991B_{2,2}$$

$$q_3 = -4.07986X^3 + 10.9524X^2 - 7.65997X + 1.00645$$
$$= 1.00645B_{0,3} - 1.54688B_{1,3} - 0.449405B_{2,3} + 0.218998B_{3,3}$$

$$q_4 = -0.234375X^4 - 3.61111X^3 + 10.651X^2 - 7.59301X + 1.0031$$
$$= 1.0031B_{0,4} - 0.895151B_{1,4} - 1.01823B_{2,4} - 0.268911B_{3,4} + 0.21565B_{4,4}$$



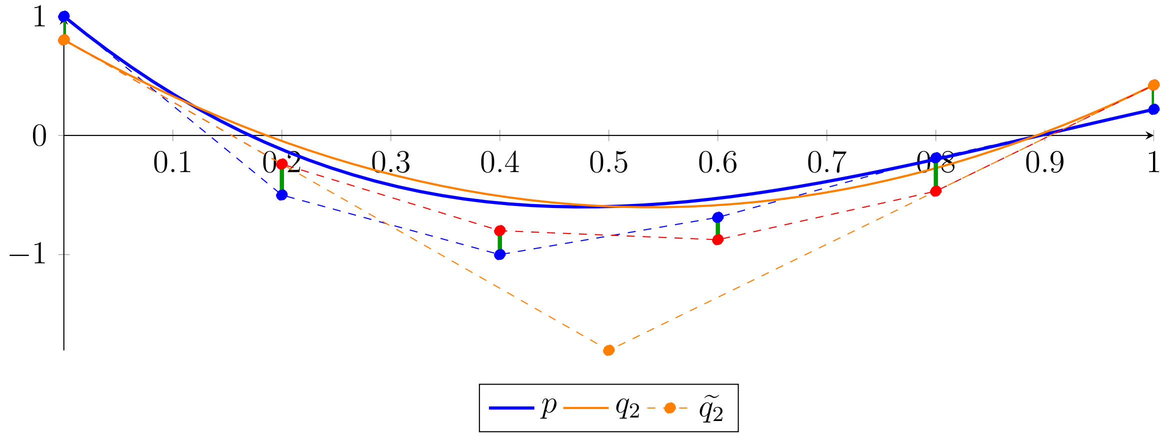
Degree reduction and raising matrices:

$$M_{5,2} = \begin{pmatrix} 0.821429 & -0.428571 & 0.107143 \\ 0.321429 & 0.285714 & -0.107143 \\ 1.249{\cdot}10^{-16} & 0.642857 & -0.142857 \\ -0.142857 & 0.642857 & 0 \\ -0.107143 & 0.285714 & 0.321429 \\ 0.107143 & -0.428571 & 0.821429 \end{pmatrix} \quad M_{2,5} = \begin{pmatrix} 1 & 0.6 & 0.3 & 0.1 & 2.17604 \\ 8.10463{\cdot}10^{-15} & 0.4 & 0.6 & 0.6 & 0. \\ 4.996{\cdot}10^{-15} & -1.59872{\cdot}10^{-14} & 0.1 & 0.3 & 0. \end{pmatrix}$$

Degree reduction and raising:

$$\begin{aligned} q_2 &= 4.83259X^2 - 5.21205X + 0.802455 \\ &= 0.802455B_{0,2} - 1.80357B_{1,2} + 0.422991B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= 1.25711{\cdot}10^{-12}X^5 - 3.15137{\cdot}10^{-12}X^4 + 2.76557{\cdot}10^{-12}X^3 + 4.83259X^2 - 5.21205X + 0.802455 \\ &= 0.802455B_{0,5} - 0.239955B_{1,5} - 0.799107B_{2,5} - 0.875B_{3,5} - 0.467634B_{4,5} + 0.422991B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.280134$.
Bounding polynomials M and m:

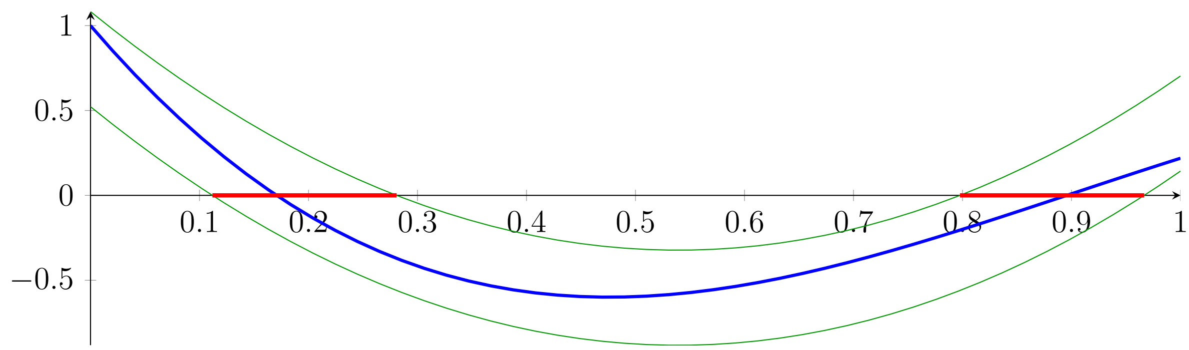
$$M = 4.83259X^2 - 5.21205X + 1.08259$$

$$m = 4.83259X^2 - 5.21205X + 0.522321$$

Root of M and m:

$$N(M) = \{0.280835, 0.797687\} \qquad N(m) = \{0.111804, 0.966718\}$$

Intersection intervals:



$[0.111804, 0.280835]$, $[0.797687, 0.966718]$

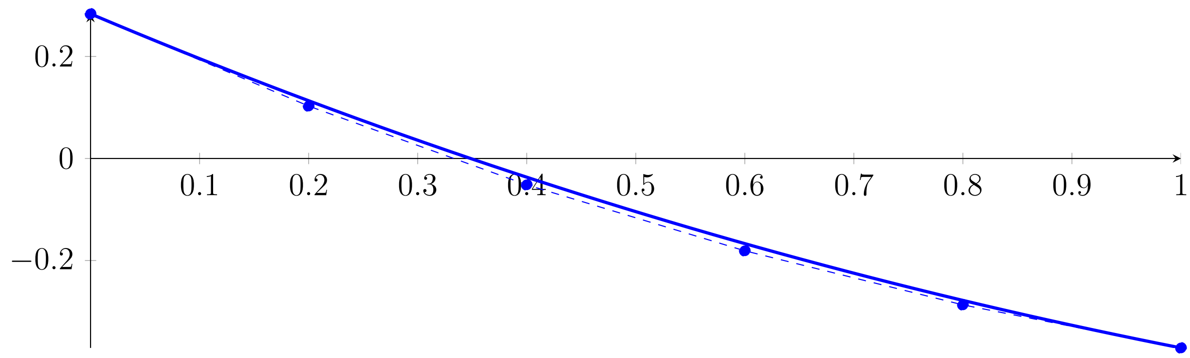
Longest intersection interval: 0.169031

$\Longrightarrow$ Selective recursion: interval 1: $[0.0559021, 0.140418]$, interval 2: $[0.398843, 0.483359]$,

2.3 Recursion Branch 1 1 1 in Interval 1: $[0.0559021, 0.140418]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 0.0001078X^5 - 0.0014292X^4 - 0.0133082X^3 + 0.26337X^2 - 0.903613X + 0.283521$$
$$= 0.283521B_{0,5}(X) + 0.102799B_{1,5}(X) - 0.0515868B_{2,5}(X)$$
$$- 0.180966B_{3,5}(X) - 0.286956B_{4,5}(X) - 0.371351B_{5,5}(X)$$



Best approximation polynomials of degree 0, 1, 2, 3 and 4:

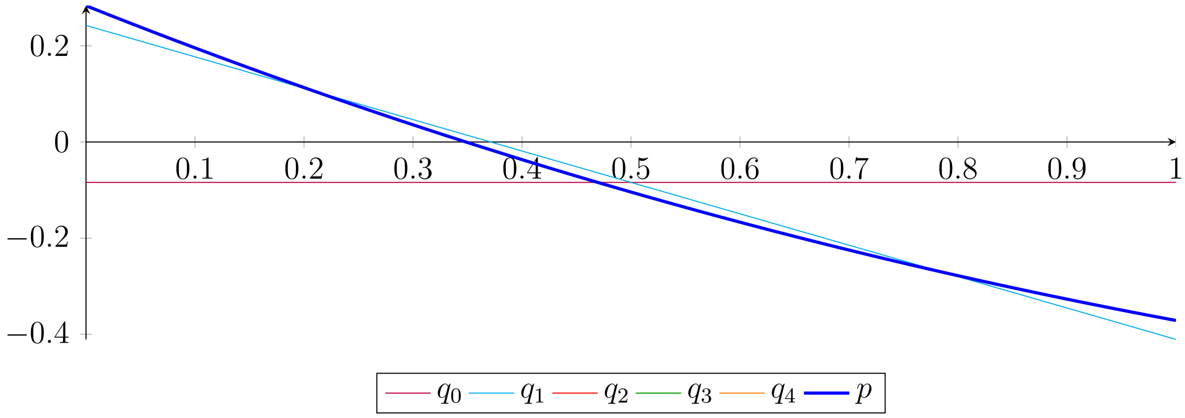
$$q_0 = -0.08409$$
$$= -0.08409B_{0,0}$$

$$q_1 = -0.653287X + 0.242553$$
$$= 0.242553B_{0,1} - 0.410733B_{1,1}$$

$$q_2 = 0.24115X^2 - 0.894437X + 0.282745$$
$$= 0.282745B_{0,2} - 0.164473B_{1,2} - 0.370542B_{2,2}$$

$$q_3 = -0.0158671X^3 + 0.264951X^2 - 0.903957X + 0.283538$$
$$= 0.283538B_{0,3} - 0.0177807B_{1,3} - 0.230783B_{2,3} - 0.371335B_{3,3}$$

$$q_4 = -0.0011597X^4 - 0.0135477X^3 + 0.26346X^2 - 0.903626X + 0.283522$$
$$= 0.283522B_{0,4} + 0.0576154B_{1,4} - 0.124381B_{2,4} - 0.265855B_{3,4} - 0.371352B_{4,4}$$



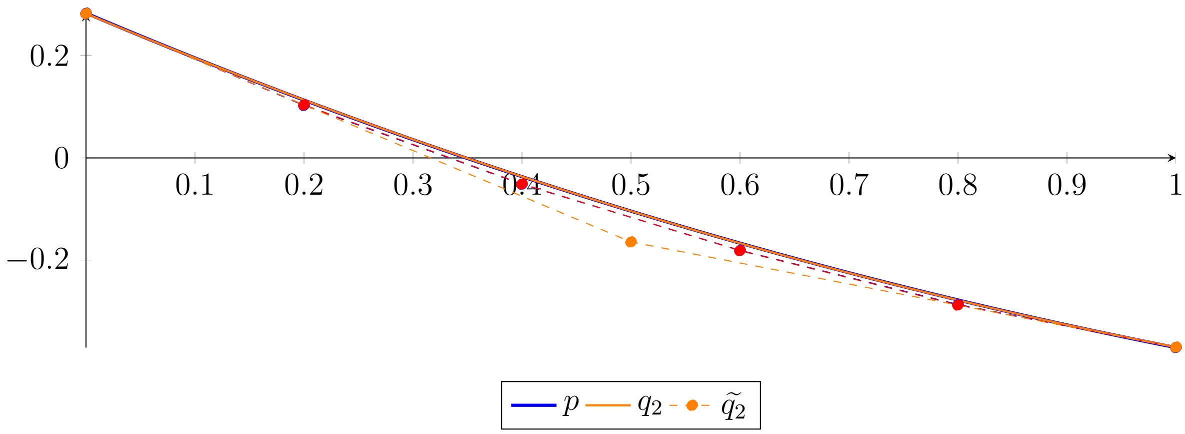
Degree reduction and raising matrices:

$$M_{5,2} = \begin{pmatrix} 0.821429 & -0.428571 & 0.107143 \\ 0.321429 & 0.285714 & -0.107143 \\ 1.249\cdot 10^{-16} & 0.642857 & -0.142857 \\ -0.142857 & 0.642857 & 0 \\ -0.107143 & 0.285714 & 0.321429 \\ 0.107143 & -0.428571 & 0.821429 \end{pmatrix} \quad M_{2,5} = \begin{pmatrix} 1 & 0.6 & 0.3 & 0.1 & 2.17604 \\ 8.10463\cdot 10^{-15} & 0.4 & 0.6 & 0.6 & 0. \\ 4.996\cdot 10^{-15} & -1.59872\cdot 10^{-14} & 0.1 & 0.3 & 0. \end{pmatrix}$$

Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.24115X^2 - 0.894437X + 0.282745 \\ &= 0.282745B_{0,2} - 0.164473B_{1,2} - 0.370542B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 5.04929\cdot 10^{-13}X^5 - 1.11688\cdot 10^{-12}X^4 + 8.40994\cdot 10^{-13}X^3 + 0.24115X^2 - 0.894437X + 0.282745 \\ &= 0.282745B_{0,5} + 0.103858B_{1,5} - 0.0509147B_{2,5} - 0.181572B_{3,5} - 0.288114B_{4,5} - 0.370542B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00115826$.
Bounding polynomials M and m:

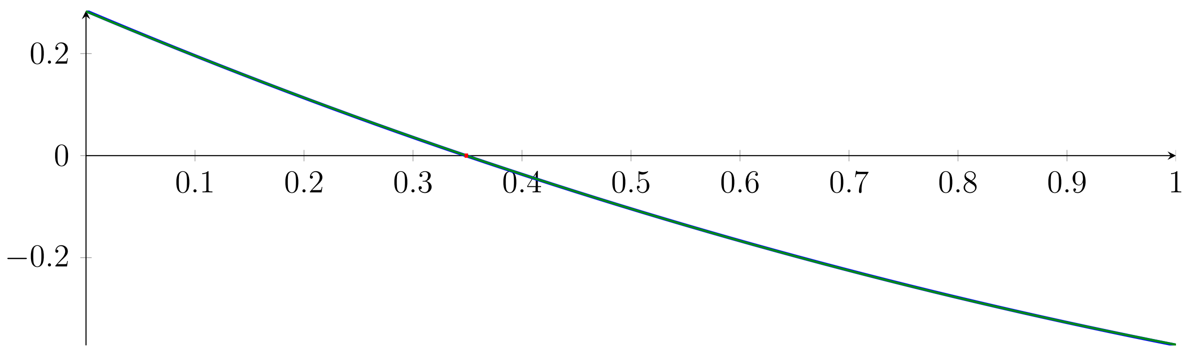
$$M = 0.24115X^2 - 0.894437X + 0.283903$$

$$m = 0.24115X^2 - 0.894437X + 0.281587$$

Root of M and m:

$$N(M) = \{0.350539, 3.3585\} \qquad N(m) = \{0.347349, 3.36169\}$$

Intersection intervals:



$[0.347349, 0.350539]$

Longest intersection interval: 0.00319018
$\Longrightarrow$ Selective recursion: interval 1: $[0.0852585, 0.0855281]$,

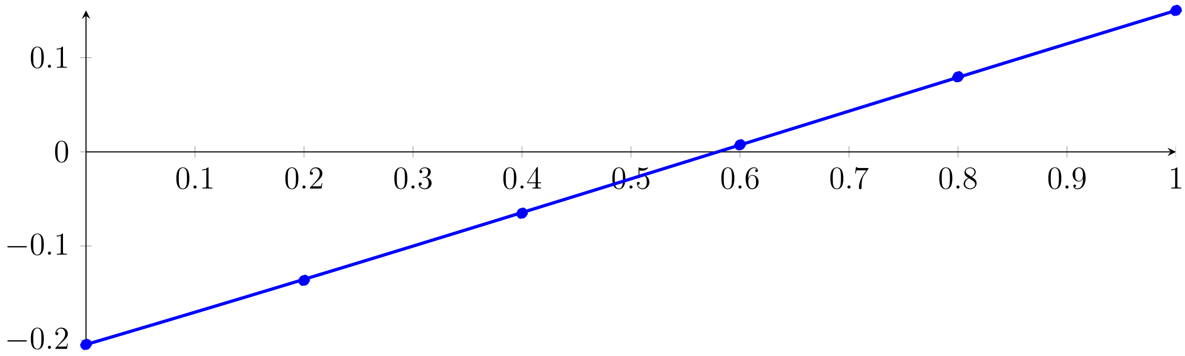
2.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.0852585, 0.0855281]$

Found root in interval $[0.0852585, 0.0855281]$ at recursion depth 4!

2.5 Recursion Branch 1 1 2 in Interval 2: $[0.398843, 0.483359]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.0001078X^5 + 0.00075793X^4 - 0.0187558X^3 + 0.0321977X^2 + 0.340572X - 0.204667 \\ &= -0.204667B_{0,5}(X) - 0.136552B_{1,5}(X) - 0.0652183B_{2,5}(X) \\ &\quad + 0.00746004B_{3,5}(X) + 0.0797585B_{4,5}(X) + 0.150213B_{5,5}(X) \end{aligned}$$



Best approximation polynomials of degree 0, 1, 2, 3 and 4:

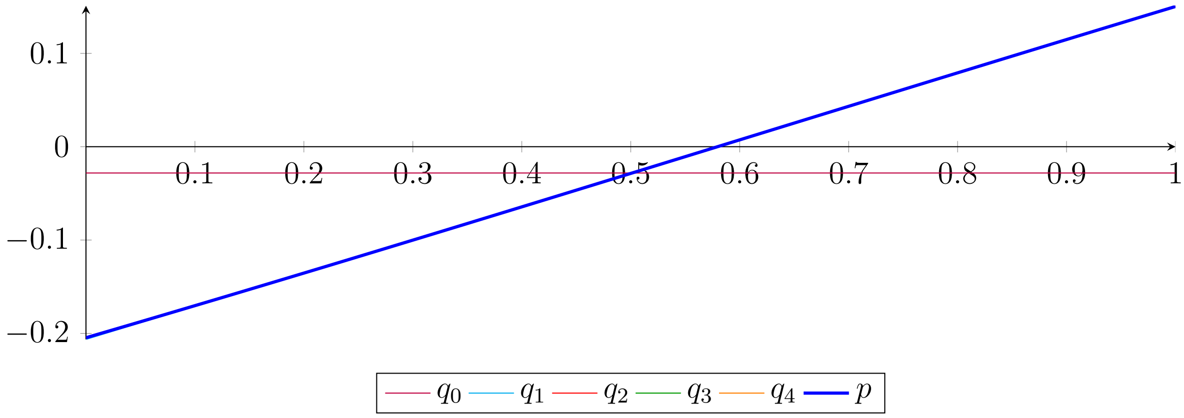
$$\begin{aligned} q_0 &= -0.0281677 \\ &= -0.0281677B_{0,0} \end{aligned}$$

$$\begin{aligned} q_1 &= 0.356573X - 0.206454 \\ &= -0.206454B_{0,1} + 0.150119B_{1,1} \end{aligned}$$

$$\begin{aligned} q_2 &= 0.00555581X^2 + 0.351017X - 0.205528 \\ &= -0.205528B_{0,2} - 0.0300196B_{1,2} + 0.151045B_{2,2} \end{aligned}$$

$$\begin{aligned} q_3 &= -0.0169405X^3 + 0.0309666X^2 + 0.340852X - 0.204681 \\ &= -0.204681B_{0,3} - 0.0910635B_{1,3} + 0.0328762B_{2,3} + 0.150198B_{3,3} \end{aligned}$$

$$\begin{aligned} q_4 &= 0.00102743X^4 - 0.0189954X^3 + 0.0322876X^2 + 0.340559X - 0.204666 \\ &= -0.204666B_{0,4} - 0.119527B_{1,4} - 0.0290056B_{2,4} + 0.0621478B_{3,4} + 0.150212B_{4,4} \end{aligned}$$



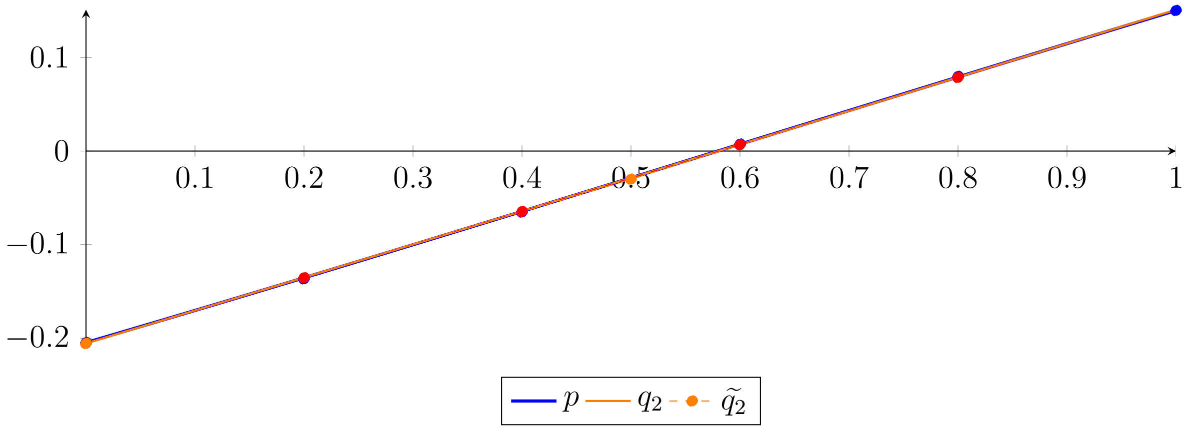
Degree reduction and raising matrices:

$$M_{5,2} = \begin{pmatrix} 0.821429 & -0.428571 & 0.107143 \\ 0.321429 & 0.285714 & -0.107143 \\ 1.249\cdot 10^{-16} & 0.642857 & -0.142857 \\ -0.142857 & 0.642857 & 0 \\ -0.107143 & 0.285714 & 0.321429 \\ 0.107143 & -0.428571 & 0.821429 \end{pmatrix} \quad M_{2,5} = \begin{pmatrix} 1 & 0.6 & 0.3 & 0.1 & 2.1760 \\ 8.10463\cdot 10^{-15} & 0.4 & 0.6 & 0.6 & 0 \\ 4.996\cdot 10^{-15} & -1.59872\cdot 10^{-14} & 0.1 & 0.3 & 0 \end{pmatrix}$$

Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.00555581X^2 + 0.351017X - 0.205528 \\ &= -0.205528B_{0,2} - 0.0300196B_{1,2} + 0.151045B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 5.67324\cdot 10^{-14}X^5 - 1.83464\cdot 10^{-13}X^4 + 1.99285\cdot 10^{-13}X^3 + 0.00555581X^2 + 0.351017X - 0.205528 \\ &= -0.205528B_{0,5} - 0.135325B_{1,5} - 0.0645657B_{2,5} + 0.00674878B_{3,5} + 0.0786189B_{4,5} + 0.151045B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.00122773$.
Bounding polynomials M and m:

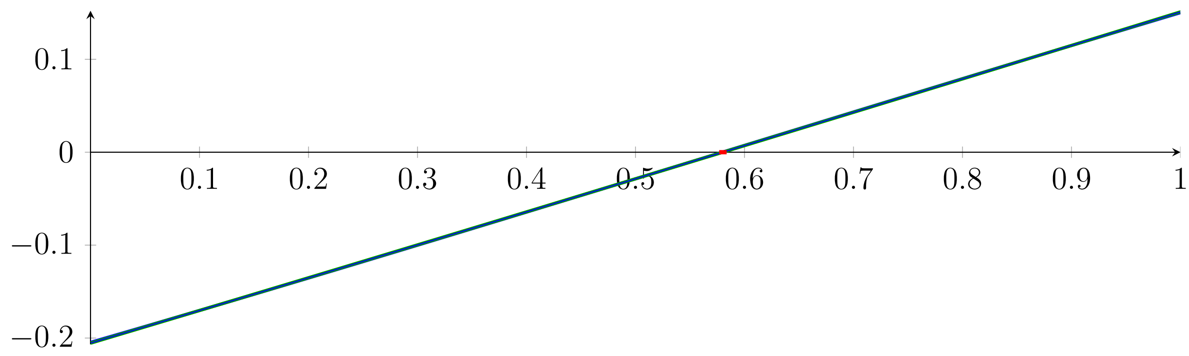
$$M = 0.00555581X^2 + 0.351017X - 0.2043$$

$$m = 0.00555581X^2 + 0.351017X - 0.206756$$

Root of M and m:

$$N(M) = \{-63.7569, 0.576759\} \qquad N(m) = \{-63.7637, 0.583628\}$$

Intersection intervals:



$[0.576759, 0.583628]$

Longest intersection interval: 0.00686912
$\Longrightarrow$ Selective recursion: interval 1: $[0.447588, 0.448169]$,

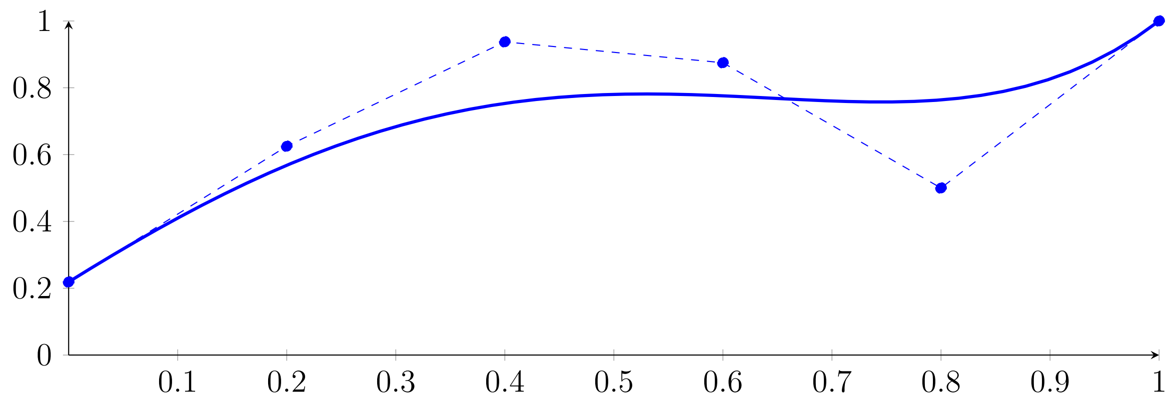
2.6 Recursion Branch 1 1 2 1 in Interval 1: $[0.447588, 0.448169]$

Found root in interval $[0.447588, 0.448169]$ at recursion depth 4!

2.7 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.78125X^5 + 1.71875X^4 - 2.8125X^3 - 0.9375X^2 + 2.03125X + 0.21875 \\ &= 0.21875B_{0,5}(X) + 0.625B_{1,5}(X) + 0.9375B_{2,5}(X) + 0.875B_{3,5}(X) + 0.5B_{4,5}(X) + 1B_{5,5}(X) \end{aligned}$$



Best approximation polynomials of degree 0, 1, 2, 3 and 4:

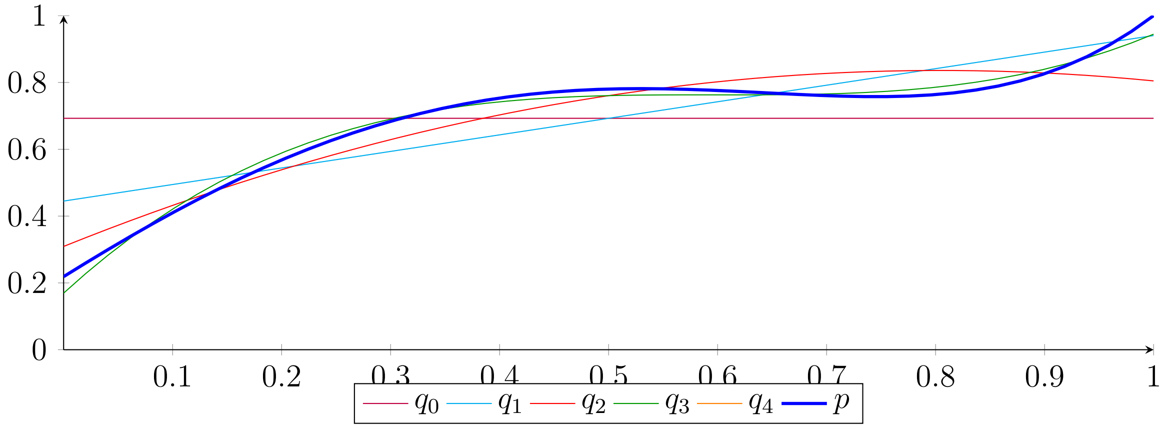
$$\begin{aligned} q_0 &= 0.692708 \\ &= 0.692708B_{0,0} \end{aligned}$$

$$\begin{aligned} q_1 &= 0.495536X + 0.44494 \\ &= 0.44494B_{0,1} + 0.940476B_{1,1} \end{aligned}$$

$$\begin{aligned} q_2 &= -0.814732X^2 + 1.31027X + 0.309152 \\ &= 0.309152B_{0,2} + 0.964286B_{1,2} + 0.804688B_{2,2} \end{aligned}$$

$$\begin{aligned} q_3 &= 2.79514X^3 - 5.00744X^2 + 2.98735X + 0.169395 \\ &= 0.169395B_{0,3} + 1.16518B_{1,3} + 0.491815B_{2,3} + 0.944444B_{3,3} \end{aligned}$$

$$\begin{aligned} q_4 &= 3.67187X^4 - 4.54861X^3 - 0.286458X^2 + 1.93824X + 0.22185 \\ &= 0.22185B_{0,4} + 0.706411B_{1,4} + 1.14323B_{2,4} + 0.395151B_{3,4} + 0.9969B_{4,4} \end{aligned}$$



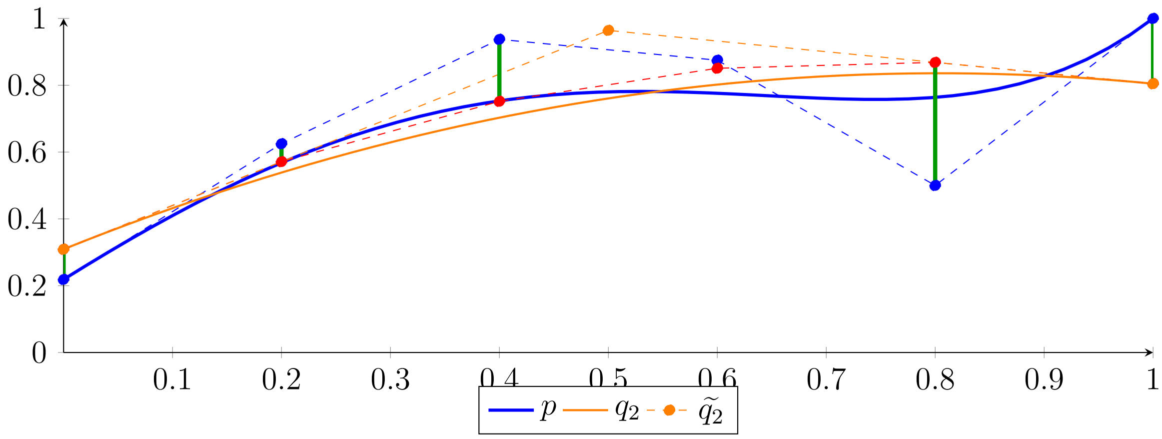
Degree reduction and raising matrices:

$$M_{5,2} = \begin{pmatrix} 0.821429 & -0.428571 & 0.107143 \\ 0.321429 & 0.285714 & -0.107143 \\ 1.249\cdot 10^{-16} & 0.642857 & -0.142857 \\ -0.142857 & 0.642857 & 0 \\ -0.107143 & 0.285714 & 0.321429 \\ 0.107143 & -0.428571 & 0.821429 \end{pmatrix} \quad M_{2,5} = \begin{pmatrix} 1 & 0.6 & 0.3 & 0.1 & 2.17604 \\ 8.10463\cdot 10^{-15} & 0.4 & 0.6 & 0.6 & 0 \\ 4.996\cdot 10^{-15} & -1.59872\cdot 10^{-14} & 0.1 & 0.3 & 0 \end{pmatrix}$$

Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.814732X^2 + 1.31027X + 0.309152 \\ &= 0.309152B_{0,2} + 0.964286B_{1,2} + 0.804688B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -3.16791\cdot 10^{-12}X^5 + 7.47069\cdot 10^{-12}X^4 - 6.11289\cdot 10^{-12}X^3 - 0.814732X^2 + 1.31027X + 0.309152 \\ &= 0.309152B_{0,5} + 0.571205B_{1,5} + 0.751786B_{2,5} + 0.850893B_{3,5} + 0.868527B_{4,5} + 0.804688B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.368527$.
Bounding polynomials M and m:

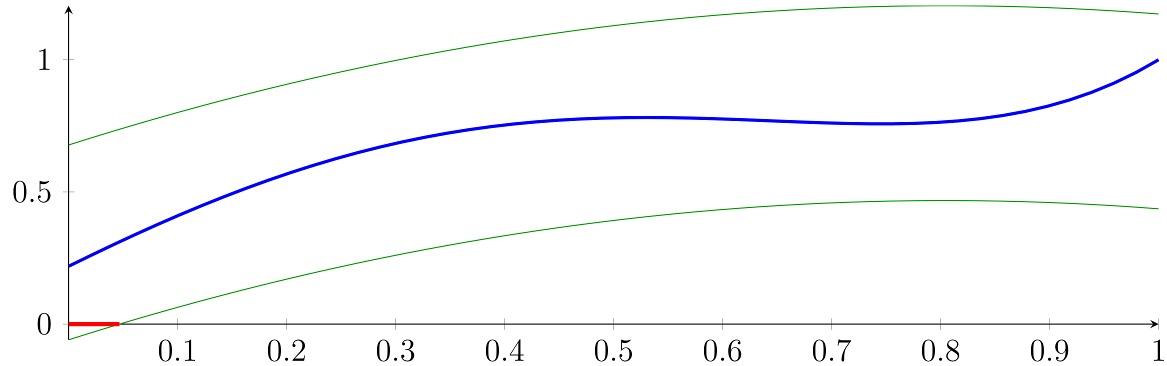
$$M = -0.814732X^2 + 1.31027X + 0.677679$$

$$m = -0.814732X^2 + 1.31027X - 0.059375$$

Root of M and m:

$$N(M) = \{-0.411774, 2.01999\} \qquad N(m) = \{0.0466695, 1.56155\}$$

Intersection intervals:



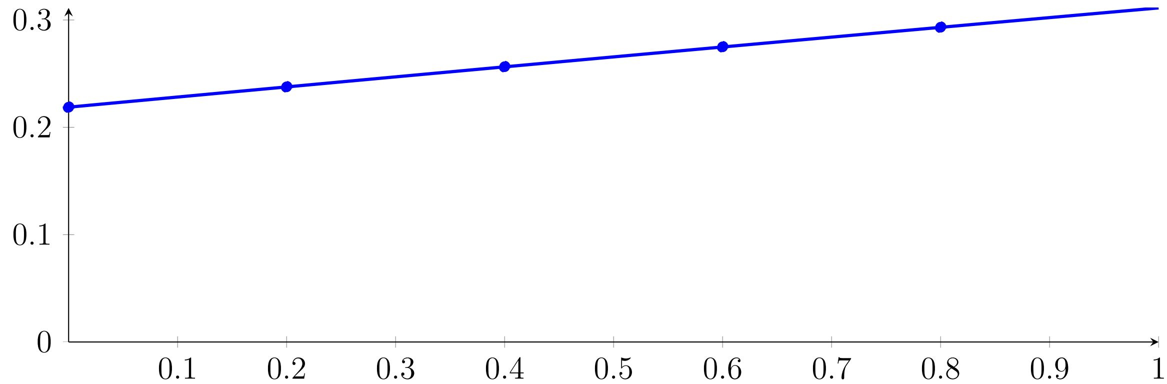
$[0, 0.0466695]$

Longest intersection interval: 0.0466695
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.523335]$,

2.8 Recursion Branch 1 2 1 in Interval 1: $[0.5, 0.523335]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 1.72964 \cdot 10^{-07} X^5 + 8.15351 \cdot 10^{-06} X^4 - 0.000285885 X^3 - 0.00204191 X^2 + 0.0947974 X + 0.21875$$
$$= 0.21875 B_{0,5}(X) + 0.237709 B_{1,5}(X) + 0.256465 B_{2,5}(X)$$
$$+ 0.274987 B_{3,5}(X) + 0.29325 B_{4,5}(X) + 0.311228 B_{5,5}(X)$$



Best approximation polynomials of degree 0, 1, 2, 3 and 4:

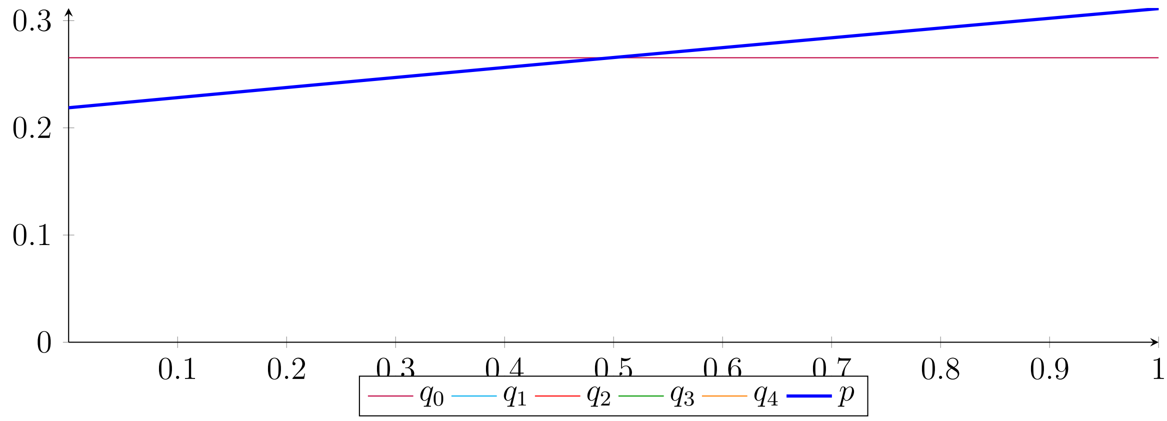
$$q_0 = 0.265398$$
$$= 0.265398 B_{0,0}$$

$$q_1 = 0.0925048 X + 0.219146$$
$$= 0.219146 B_{0,1} + 0.311651 B_{1,1}$$

$$q_2 = -0.00245645 X^2 + 0.0949613 X + 0.218736$$
$$= 0.218736 B_{0,2} + 0.266217 B_{1,2} + 0.311241 B_{2,2}$$

$$q_3 = -0.000269098 X^3 - 0.00205281 X^2 + 0.0947998 X + 0.21875$$
$$= 0.21875 B_{0,3} + 0.25035 B_{1,3} + 0.281265 B_{2,3} + 0.311228 B_{3,3}$$

$$q_4 = 8.58592 \cdot 10^{-06} X^4 - 0.000286269 X^3 - 0.00204177 X^2 + 0.0947974 X + 0.21875$$
$$= 0.21875 B_{0,4} + 0.242449 B_{1,4} + 0.265808 B_{2,4} + 0.288756 B_{3,4} + 0.311228 B_{4,4}$$



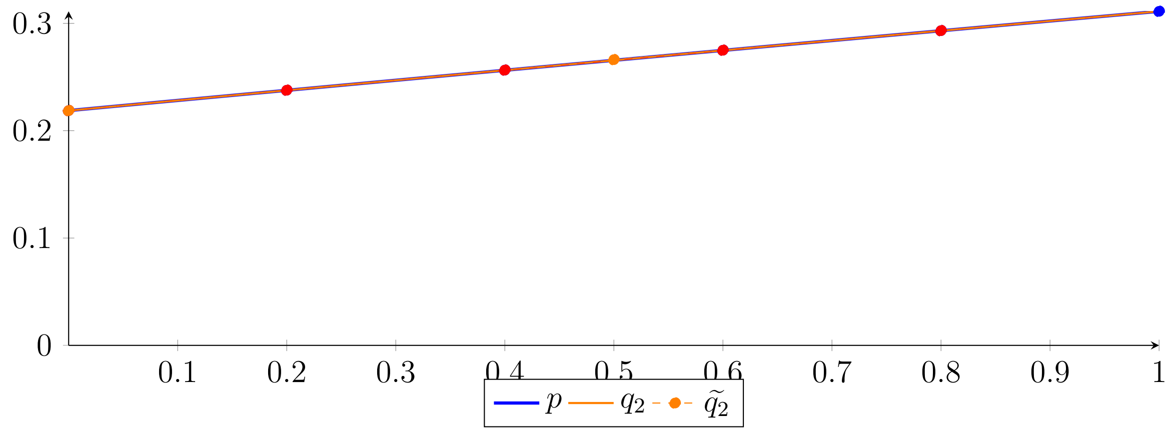
Degree reduction and raising matrices:

$$M_{5,2} = \begin{pmatrix} 0.821429 & -0.428571 & 0.107143 \\ 0.321429 & 0.285714 & -0.107143 \\ 1.249\cdot10^{-16} & 0.642857 & -0.142857 \\ -0.142857 & 0.642857 & 0 \\ -0.107143 & 0.285714 & 0.321429 \\ 0.107143 & -0.428571 & 0.821429 \end{pmatrix} \quad M_{2,5} = \begin{pmatrix} 1 & 0.6 & 0.3 & 0.1 & 2.1760 \\ 8.10463\cdot10^{-15} & 0.4 & 0.6 & 0.6 & 0 \\ 4.996\cdot10^{-15} & -1.59872\cdot10^{-14} & 0.1 & 0.3 & 0. \end{pmatrix}$$

Degree reduction and raising:

$$q_2 = -0.00245645X^2 + 0.0949613X + 0.218736$$
$$= 0.218736B_{0,2} + 0.266217B_{1,2} + 0.311241B_{2,2}$$

$$\widetilde{q}_2 = -1.17323\cdot10^{-12}X^5 + 2.76473\cdot10^{-12}X^4 - 2.26041\cdot10^{-12}X^3 - 0.00245645X^2 + 0.0949613X + 0.218736$$
$$= 0.218736B_{0,5} + 0.237729B_{1,5} + 0.256475B_{2,5} + 0.274976B_{3,5} + 0.293232B_{4,5} + 0.311241B_{5,5}$$



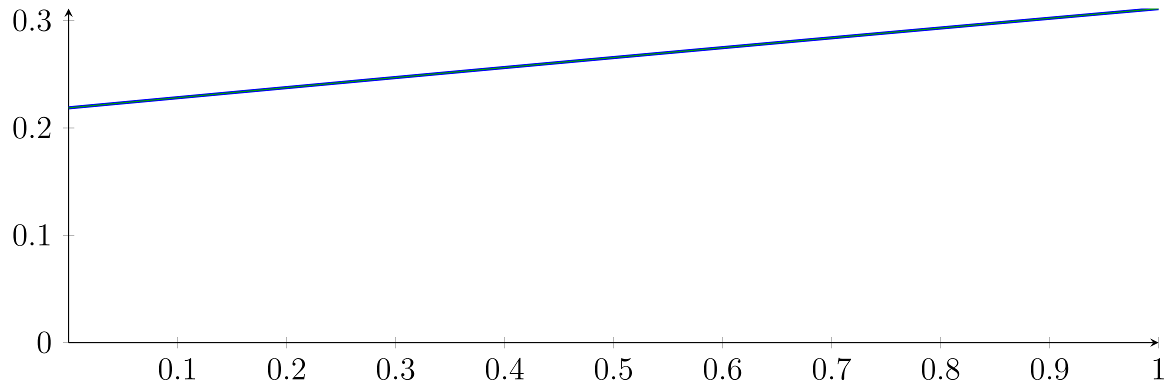
The maximum difference of the Bézier coefficients is $\delta = 1.92014\cdot10^{-05}$.
Bounding polynomials M and m:

$$M = -0.00245645X^2 + 0.0949613X + 0.218756$$
$$m = -0.00245645X^2 + 0.0949613X + 0.218717$$

Root of M and m:

$$N(M) = \{-2.18062, 40.8385\} \qquad N(m) = \{-2.18026, 40.8381\}$$

Intersection intervals:

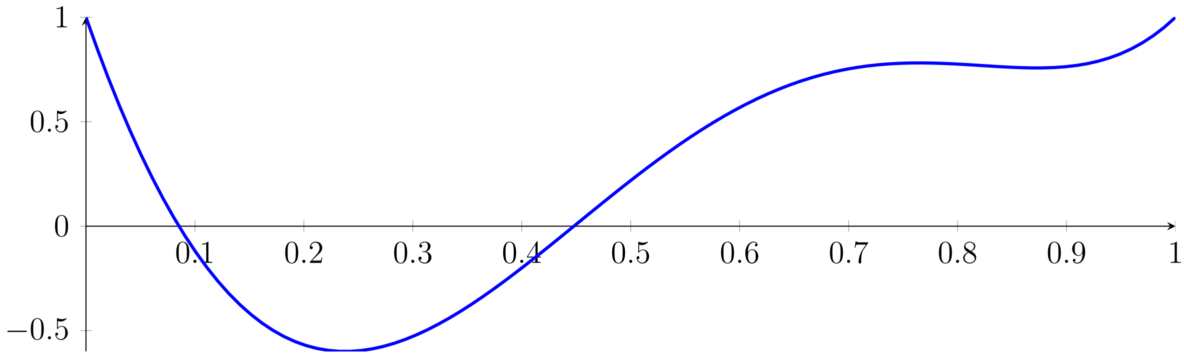


No intersection intervals with the x axis.

2.9 Result: 2 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = 25X^5 - 35X^4 - 15X^3 + 40X^2 - 15X + 1$$



Result: Root Intervals

$$[0.0852585, 0.0855281],\ [0.447588, 0.448169]$$

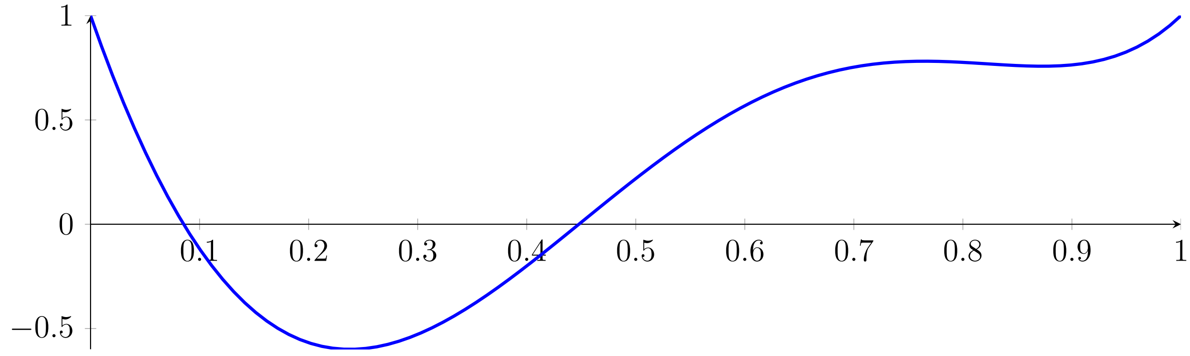
with precision $\varepsilon = 0.001$.

3 `CubeClip` Applied to a Polynomial of 5th Degree with Two Roots

$$25X^5 - 35X^4 - 15X^3 + 40X^2 - 15X + 1$$

Called `CubeClip` with input polynomial on interval $[0, 1]$:

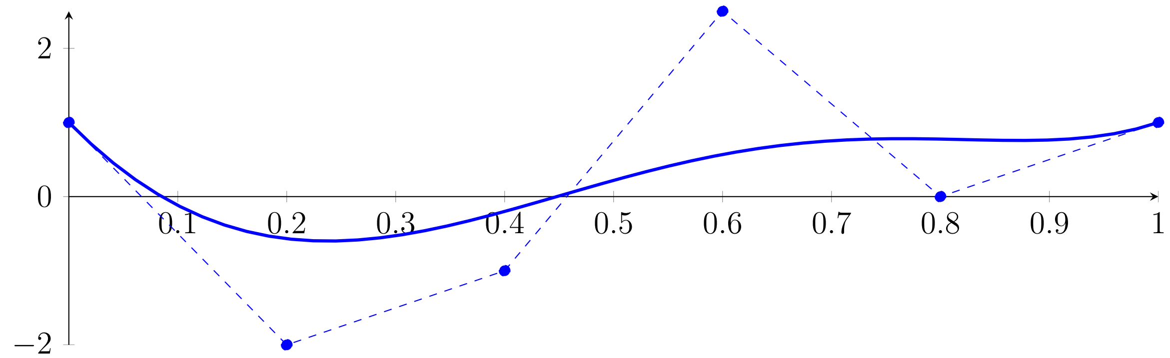
$$p = 25X^5 - 35X^4 - 15X^3 + 40X^2 - 15X + 1$$



3.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 25X^5 - 35X^4 - 15X^3 + 40X^2 - 15X + 1 \\ &= 1B_{0,5}(X) - 2B_{1,5}(X) - 1B_{2,5}(X) + 2.5B_{3,5}(X) + 0B_{4,5}(X) + 1B_{5,5}(X) \end{aligned}$$



Best approximation polynomials of degree 0, 1, 2, 3 and 4:

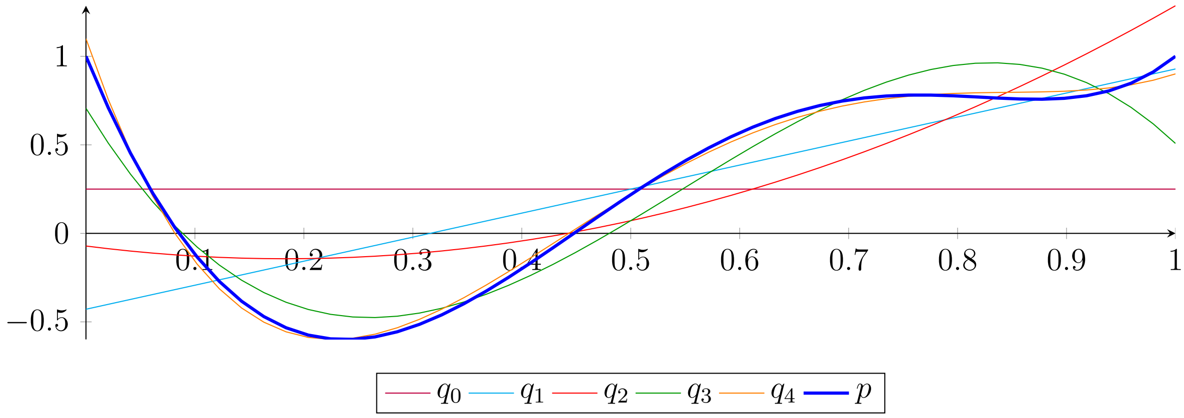
$$\begin{aligned} q_0 &= 0.25 \\ &= 0.25B_{0,0} \end{aligned}$$

$$\begin{aligned} q_1 &= 1.35714X - 0.428571 \\ &= -0.428571B_{0,1} + 0.928571B_{1,1} \end{aligned}$$

$$\begin{aligned} q_2 &= 2.14286X^2 - 0.785714X - 0.0714286 \\ &= -0.0714286B_{0,2} - 0.464286B_{1,2} + 1.28571B_{2,2} \end{aligned}$$

$$\begin{aligned} q_3 &= -15.5556X^3 + 25.4762X^2 - 10.119X + 0.706349 \\ &= 0.706349B_{0,3} - 2.66667B_{1,3} + 2.45238B_{2,3} + 0.507937B_{3,3} \end{aligned}$$

$$\begin{aligned} q_4 &= 27.5X^4 - 70.5556X^3 + 60.8333X^2 - 17.9762X + 1.09921 \\ &= 1.09921B_{0,4} - 3.39484B_{1,4} + 2.25B_{2,4} + 0.394841B_{3,4} + 0.900794B_{4,4} \end{aligned}$$



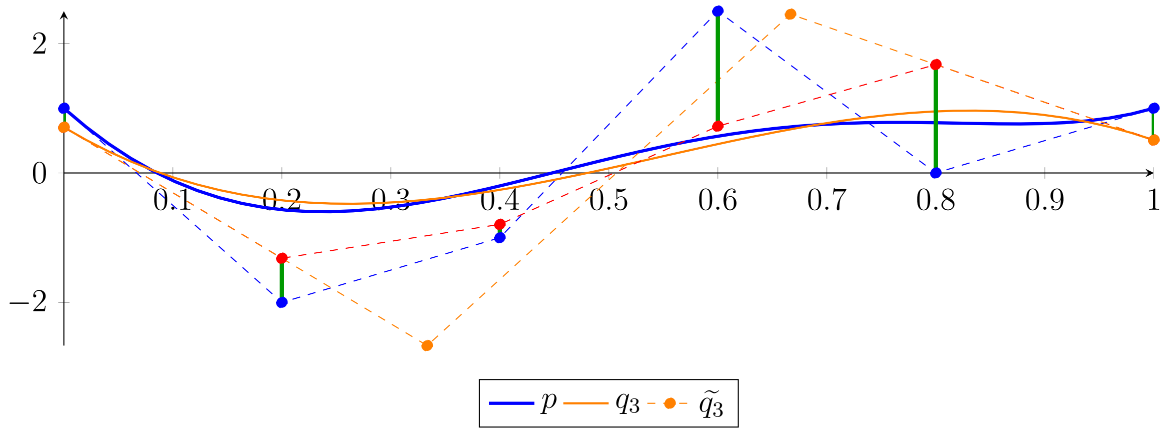
Degree reduction and raising matrices:

$$M_{5,3} = \begin{pmatrix} 0.960317 & -0.428571 & 0.166667 & -0.031746 \\ 0.126984 & 0.880952 & -0.428571 & 0.0873016 \\ -0.111111 & 0.761905 & 0.047619 & -0.031746 \\ -0.031746 & 0.047619 & 0.761905 & -0.111111 \\ 0.0873016 & -0.428571 & 0.880952 & 0.126984 \\ -0.031746 & 0.166667 & -0.428571 & 0.960317 \end{pmatrix} \quad M_{3,5} = \begin{pmatrix} 1 & 0.4 & \\ 1.97065\cdot 10^{-15} & 0.6 & \\ 3.08087\cdot 10^{-15} & -1.9984\cdot 10^{-15} & \\ -1.55431\cdot 10^{-15} & 1.77636\cdot 10^{-14} & -3.3 \end{pmatrix}$$

Degree reduction and raising:

$$\begin{aligned} q_3 &= -15.5556X^3 + 25.4762X^2 - 10.119X + 0.706349 \\ &= 0.706349B_{0,3} - 2.66667B_{1,3} + 2.45238B_{2,3} + 0.507937B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -5.82645\cdot 10^{-13}X^5 + 1.75415\cdot 10^{-12}X^4 - 15.5556X^3 + 25.4762X^2 - 10.119X + 0.706349 \\ &= 0.706349B_{0,5} - 1.31746B_{1,5} - 0.793651B_{2,5} + 0.722222B_{3,5} + 1.6746B_{4,5} + 0.507937B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.77778$.
Bounding polynomials M and m:

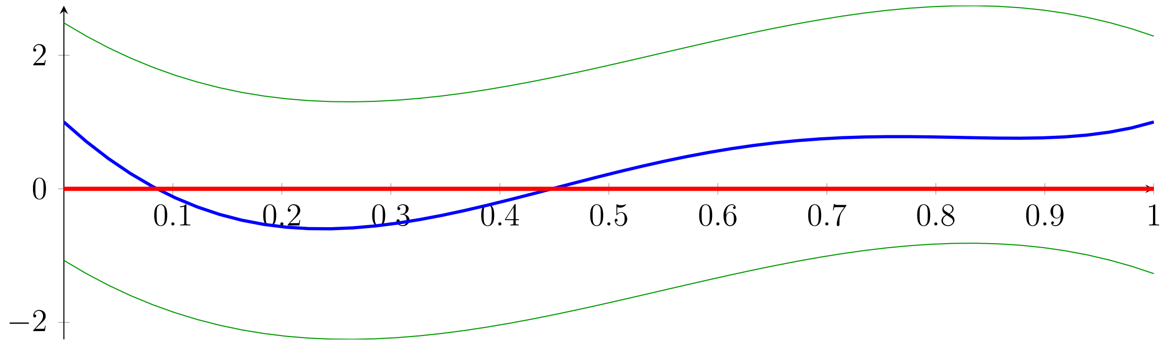
$$M = -15.5556X^3 + 25.4762X^2 - 10.119X + 2.48413$$

$$m = -15.5556X^3 + 25.4762X^2 - 10.119X - 1.07143$$

Root of M and m:

$$N(M) = \{1.20894\} \qquad N(m) = \{-0.0861935\}$$

Intersection intervals:



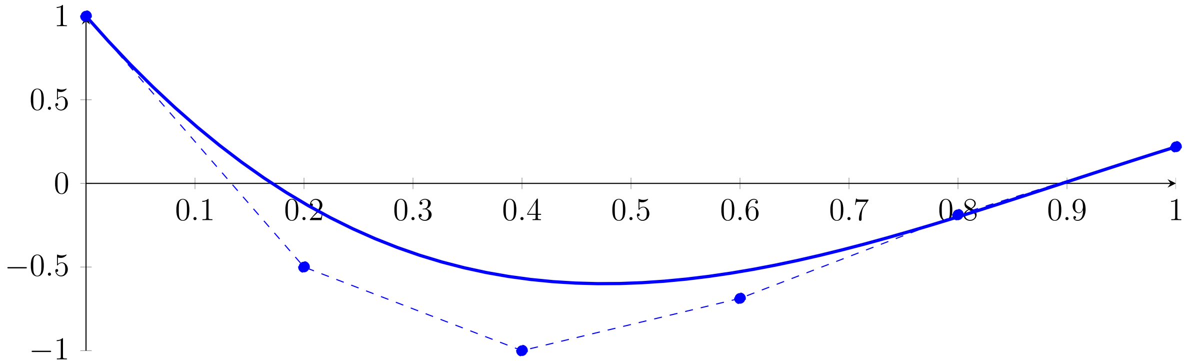
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

3.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 0.78125X^5 - 2.1875X^4 - 1.875X^3 + 10X^2 - 7.5X + 1$$
$$= 1B_{0,5}(X) - 0.5B_{1,5}(X) - 1B_{2,5}(X) - 0.6875B_{3,5}(X) - 0.1875B_{4,5}(X) + 0.21875B_{5,5}(X)$$



Best approximation polynomials of degree 0, 1, 2, 3 and 4:

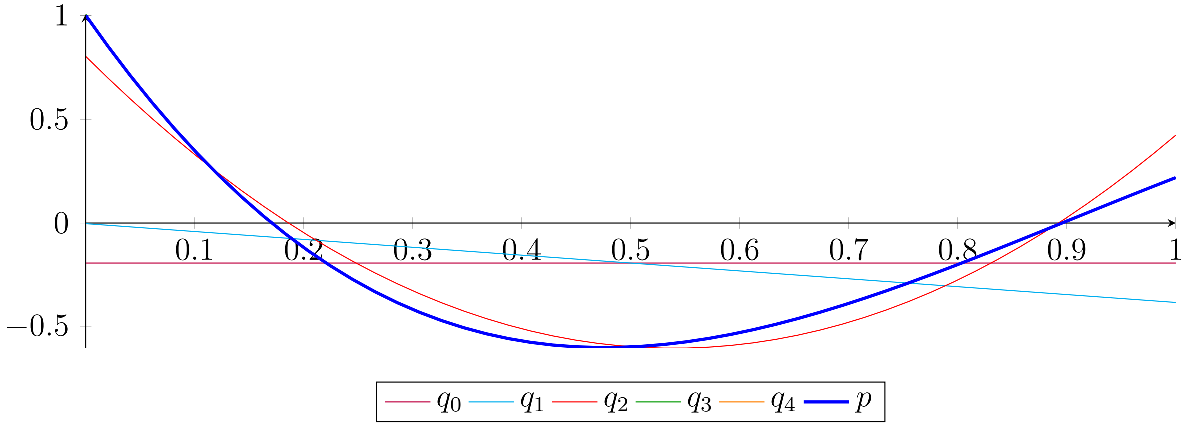
$$q_0 = -0.192708$$
$$= -0.192708B_{0,0}$$

$$q_1 = -0.379464X - 0.00297619$$
$$= -0.00297619B_{0,1} - 0.38244B_{1,1}$$

$$q_2 = 4.83259X^2 - 5.21205X + 0.802455$$
$$= 0.802455B_{0,2} - 1.80357B_{1,2} + 0.422991B_{2,2}$$

$$q_3 = -4.07986X^3 + 10.9524X^2 - 7.65997X + 1.00645$$
$$= 1.00645B_{0,3} - 1.54688B_{1,3} - 0.449405B_{2,3} + 0.218998B_{3,3}$$

$$q_4 = -0.234375X^4 - 3.61111X^3 + 10.651X^2 - 7.59301X + 1.0031$$
$$= 1.0031B_{0,4} - 0.895151B_{1,4} - 1.01823B_{2,4} - 0.268911B_{3,4} + 0.21565B_{4,4}$$



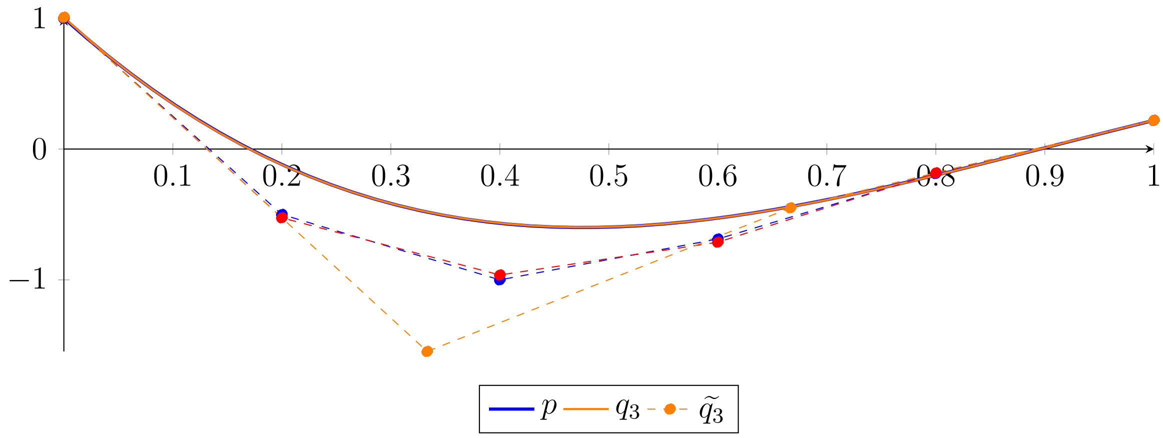
Degree reduction and raising matrices:

$$M_{5,3} = \begin{pmatrix} 0.960317 & -0.428571 & 0.166667 & -0.031746 \\ 0.126984 & 0.880952 & -0.428571 & 0.0873016 \\ -0.111111 & 0.761905 & 0.047619 & -0.031746 \\ -0.031746 & 0.047619 & 0.761905 & -0.111111 \\ 0.0873016 & -0.428571 & 0.880952 & 0.126984 \\ -0.031746 & 0.166667 & -0.428571 & 0.960317 \end{pmatrix} \quad M_{3,5} = \begin{pmatrix} 1 & 0.4 & \\ 1.97065\cdot 10^{-15} & 0.6 & \\ 3.08087\cdot 10^{-15} & -1.9984\cdot 10^{-15} & \\ -1.55431\cdot 10^{-15} & 1.77636\cdot 10^{-14} & -3.3 \end{pmatrix}$$

Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.07986X^3 + 10.9524X^2 - 7.65997X + 1.00645 \\ &= 1.00645B_{0,3} - 1.54688B_{1,3} - 0.449405B_{2,3} + 0.218998B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1.55564\cdot 10^{-12}X^5 + 4.64073\cdot 10^{-12}X^4 - 4.07986X^3 + 10.9524X^2 - 7.65997X + 1.00645 \\ &= 1.00645B_{0,5} - 0.525546B_{1,5} - 0.962302B_{2,5} - 0.711806B_{3,5} - 0.182044B_{4,5} + 0.218998B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0376984$.
Bounding polynomials M and m:

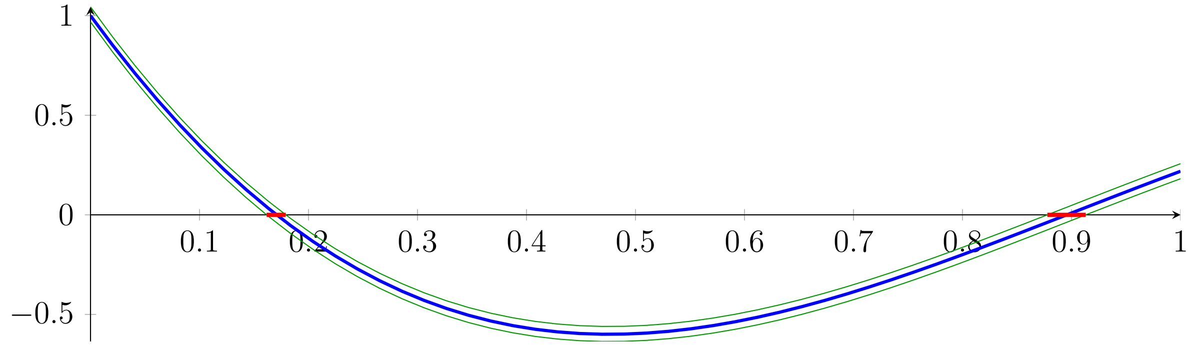
$$M = -4.07986X^3 + 10.9524X^2 - 7.65997X + 1.04415$$

$$m = -4.07986X^3 + 10.9524X^2 - 7.65997X + 0.96875$$

Root of M and m:

$$N(M) = \{0.17913, 0.877855, 1.62751\} \qquad N(m) = \{0.161532, 0.913098, 1.60987\}$$

Intersection intervals:



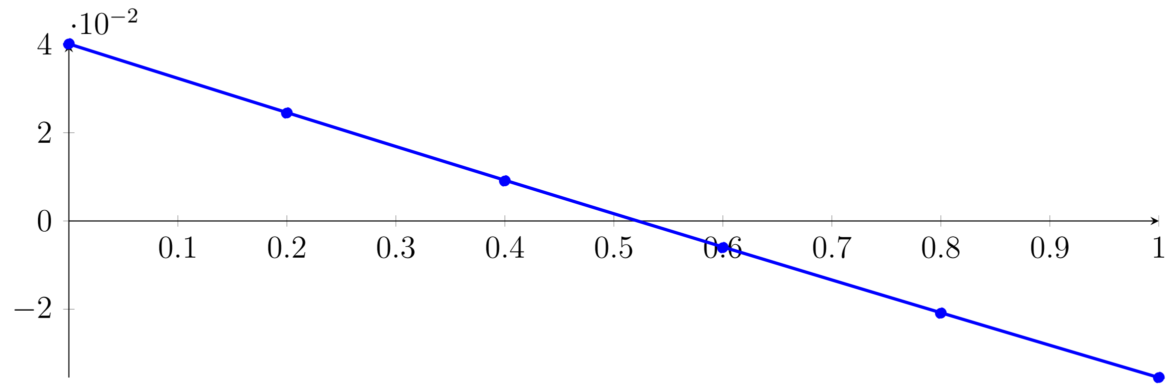
$$[0.161532, 0.17913],\ [0.877855, 0.913098]$$

Longest intersection interval: 0.0352436
$\Longrightarrow$ Selective recursion: interval 1: $[0.080766, 0.0895651]$, interval 2: $[0.438927, 0.456549]$,

3.3 Recursion Branch 1 1 1 in Interval 1: $[0.080766, 0.0895651]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.31869 \cdot 10^{-09} X^5 - 1.49292 \cdot 10^{-07} X^4 - 1.68114 \cdot 10^{-05} X^3 + 0.00271974 X^2 - 0.0783186 X + 0.0401299 \\ &= 0.0401299 B_{0,5}(X) + 0.0244662 B_{1,5}(X) + 0.00907441 B_{2,5}(X) \\ &\quad - 0.00604703 B_{3,5}(X) - 0.0208999 B_{4,5}(X) - 0.0354859 B_{5,5}(X) \end{aligned}$$



Best approximation polynomials of degree 0, 1, 2, 3 and 4:

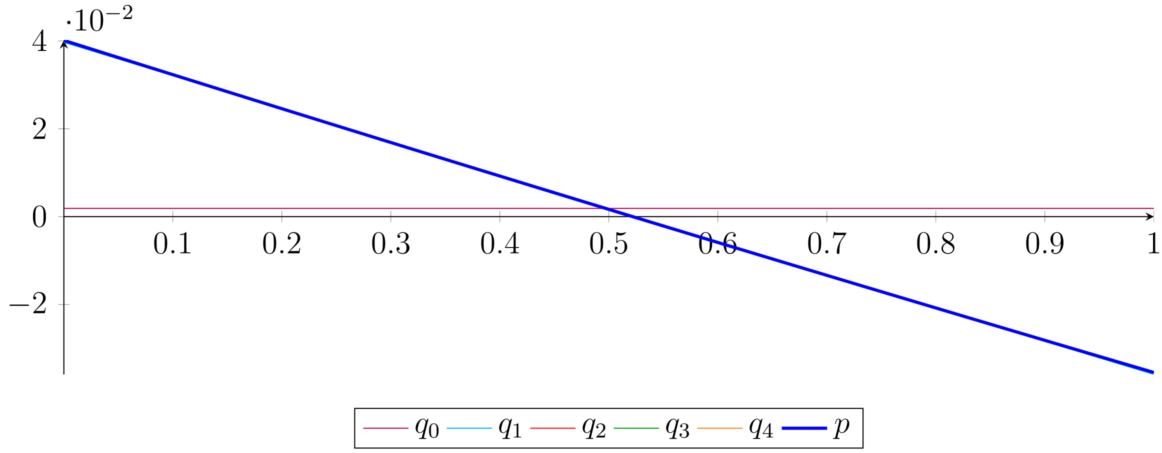
$$\begin{aligned} q_0 &= 0.00187293 \\ &= 0.00187293 B_{0,0} \end{aligned}$$

$$\begin{aligned} q_1 &= -0.0756141 X + 0.03968 \\ &= 0.03968 B_{0,1} - 0.0359341 B_{1,1} \end{aligned}$$

$$\begin{aligned} q_2 &= 0.00269427 X^2 - 0.0783083 X + 0.040129 \\ &= 0.040129 B_{0,2} + 0.000974841 B_{1,2} - 0.0354851 B_{2,2} \end{aligned}$$

$$\begin{aligned} q_3 &= -1.71063 \cdot 10^{-05} X^3 + 0.00271992 X^2 - 0.0783186 X + 0.0401299 \\ &= 0.0401299 B_{0,3} + 0.0140237 B_{1,3} - 0.0111759 B_{2,3} - 0.0354859 B_{3,3} \end{aligned}$$

$$\begin{aligned} q_4 &= -1.45995 \cdot 10^{-07} X^4 - 1.68143 \cdot 10^{-05} X^3 + 0.00271974 X^2 - 0.0783186 X + 0.0401299 \\ &= 0.0401299 B_{0,4} + 0.0205502 B_{1,4} + 0.00142387 B_{2,4} - 0.0172534 B_{3,4} - 0.0354859 B_{4,4} \end{aligned}$$



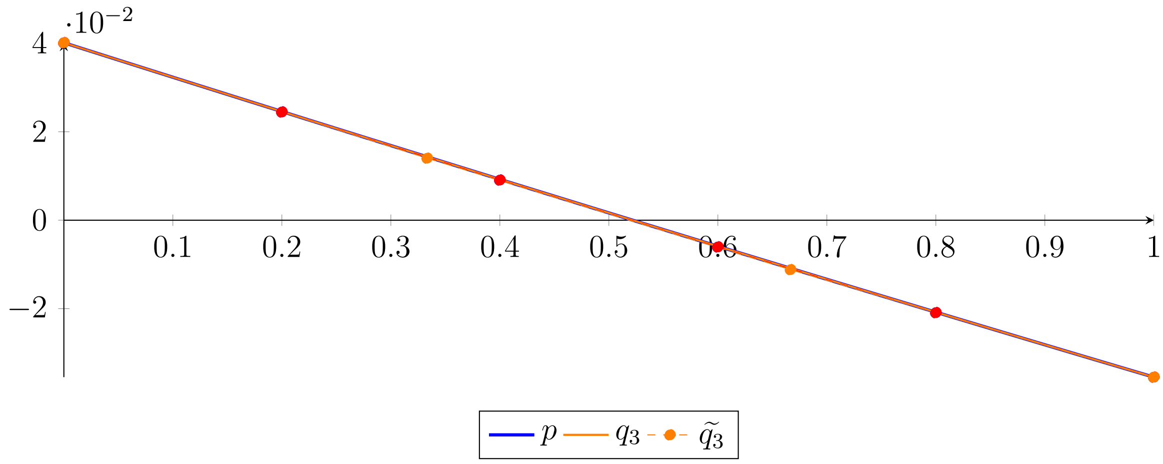
Degree reduction and raising matrices:

$$M_{5,3} = \begin{pmatrix} 0.960317 & -0.428571 & 0.166667 & -0.031746 \\ 0.126984 & 0.880952 & -0.428571 & 0.0873016 \\ -0.111111 & 0.761905 & 0.047619 & -0.031746 \\ -0.031746 & 0.047619 & 0.761905 & -0.111111 \\ 0.0873016 & -0.428571 & 0.880952 & 0.126984 \\ -0.031746 & 0.166667 & -0.428571 & 0.960317 \end{pmatrix} \quad M_{3,5} = \begin{pmatrix} 1 & 0.4 & \\ 1.97065\cdot 10^{-15} & 0.6 & \\ 3.08087\cdot 10^{-15} & -1.9984\cdot 10^{-15} & \\ -1.55431\cdot 10^{-15} & 1.77636\cdot 10^{-14} & -3.3 \end{matrix}$$

Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.71063\cdot 10^{-05}X^3 + 0.00271992X^2 - 0.0783186X + 0.0401299 \\ &= 0.0401299B_{0,3} + 0.0140237B_{1,3} - 0.0111759B_{2,3} - 0.0354859B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1.68948\cdot 10^{-13}X^5 + 4.28789\cdot 10^{-13}X^4 - 1.71063\cdot 10^{-05}X^3 \\ &\quad + 0.00271992X^2 - 0.0783186X + 0.0401299 \\ &= 0.0401299B_{0,5} + 0.0244661B_{1,5} + 0.00907442B_{2,5} \\ &\quad - 0.00604703B_{3,5} - 0.0208999B_{4,5} - 0.0354859B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.28309\cdot 10^{-09}$.

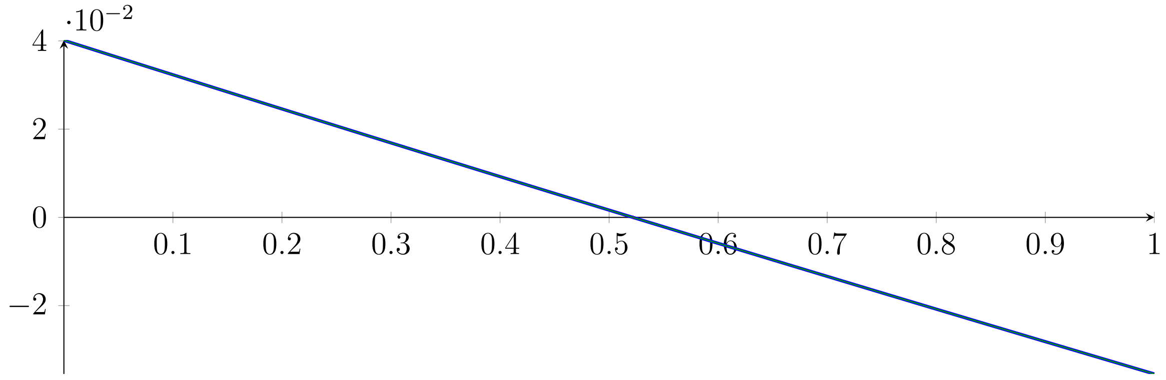
Bounding polynomials M and m:

$$M = -1.71063\cdot 10^{-05}X^3 + 0.00271992X^2 - 0.0783186X + 0.0401299$$
$$m = -1.71063\cdot 10^{-05}X^3 + 0.00271992X^2 - 0.0783186X + 0.0401299$$

Root of M and m:

$$N(M) = \{0.521818, 37.0108, 121.468\} \qquad N(m) = \{0.521818, 37.0108, 121.468\}$$

Intersection intervals:



$[0.521818, 0.521818]$

Longest intersection interval: $1.66453 \cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.0853575, 0.0853575]$,

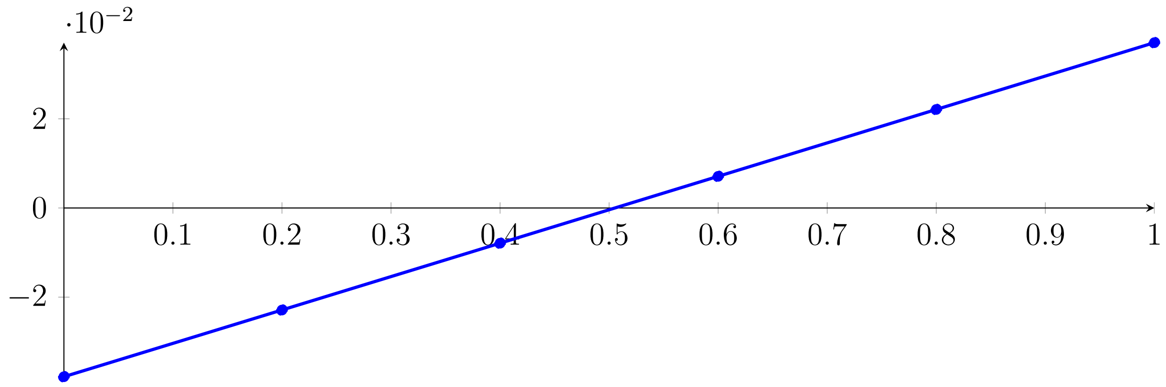
3.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.0853575, 0.0853575]$

Found root in interval $[0.0853575, 0.0853575]$ at recursion depth 4!

3.5 Recursion Branch 1 1 2 in Interval 2: $[0.438927, 0.456549]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 4.24804 \cdot 10^{-08} X^5 + 1.91561 \cdot 10^{-06} X^4 - 0.00015478 X^3 + 0.000289066 X^2 + 0.0748126 X - 0.0378581$$
$$= -0.0378581 B_{0,5}(X) - 0.0228956 B_{1,5}(X) - 0.00790421 B_{2,5}(X)$$
$$+ 0.00710064 B_{3,5}(X) + 0.0221038 B_{4,5}(X) + 0.0370907 B_{5,5}(X)$$



Best approximation polynomials of degree 0, 1, 2, 3 and 4:

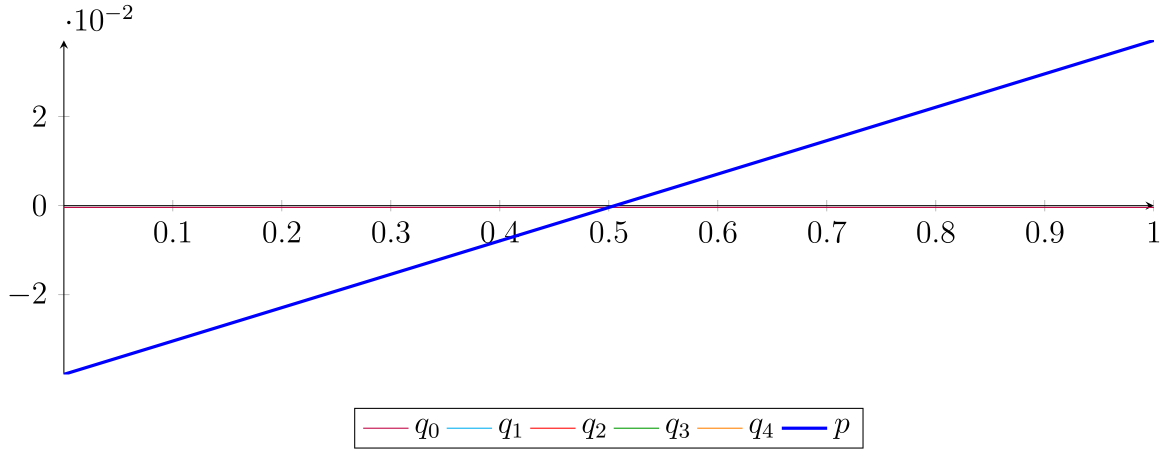
$$q_0 = -0.000393813$$
$$= -0.000393813 B_{0,0}$$

$$q_1 = 0.0749639 X - 0.0378758$$
$$= -0.0378758 B_{0,1} + 0.0370881 B_{1,1}$$

$$q_2 = 6.02563 \cdot 10^{-05} X^2 + 0.0749036 X - 0.0378657$$
$$= -0.0378657 B_{0,2} - 0.000413898 B_{1,2} + 0.0370982 B_{2,2}$$

$$q_3 = -0.00015083 X^3 + 0.000286502 X^2 + 0.0748131 X - 0.0378582$$
$$= -0.0378582 B_{0,3} - 0.0129205 B_{1,3} + 0.0121128 B_{2,3} + 0.0370906 B_{3,3}$$

$$q_4 = 2.02181 \cdot 10^{-06} X^4 - 0.000154874 X^3 + 0.000289101 X^2 + 0.0748126 X - 0.0378581$$
$$= -0.0378581 B_{0,4} - 0.019155 B_{1,4} - 0.000403682 B_{2,4} + 0.0183571 B_{3,4} + 0.0370907 B_{4,4}$$



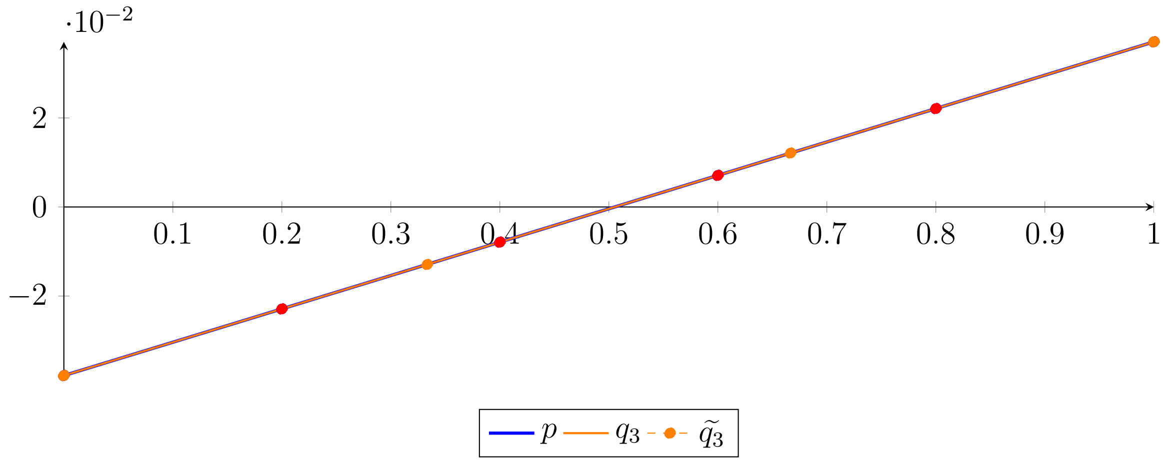
Degree reduction and raising matrices:

$$M_{5,3} = \begin{pmatrix} 0.960317 & -0.428571 & 0.166667 & -0.031746 \\ 0.126984 & 0.880952 & -0.428571 & 0.0873016 \\ -0.111111 & 0.761905 & 0.047619 & -0.031746 \\ -0.031746 & 0.047619 & 0.761905 & -0.111111 \\ 0.0873016 & -0.428571 & 0.880952 & 0.126984 \\ -0.031746 & 0.166667 & -0.428571 & 0.960317 \end{pmatrix} \quad M_{3,5} = \begin{pmatrix} 1 & 0.4 & \\ 1.97065\cdot 10^{-15} & 0.6 & \\ 3.08087\cdot 10^{-15} & -1.9984\cdot 10^{-15} & \\ -1.55431\cdot 10^{-15} & 1.77636\cdot 10^{-14} & -3.3 \end{pmatrix}$$

Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.00015083X^3 + 0.000286502X^2 + 0.0748131X - 0.0378582 \\ &= -0.0378582B_{0,3} - 0.0129205B_{1,3} + 0.0121128B_{2,3} + 0.0370906B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= 1.61572\cdot 10^{-13}X^5 - 4.10193\cdot 10^{-13}X^4 - 0.00015083X^3 + 0.000286502X^2 + 0.0748131X - 0.0378582 \\ &= -0.0378582B_{0,5} - 0.0228955B_{1,5} - 0.00790427B_{2,5} \\ &\quad + 0.00710058B_{3,5} + 0.0221039B_{4,5} + 0.0370906B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 8.7492\cdot 10^{-08}$.

Bounding polynomials M and m:

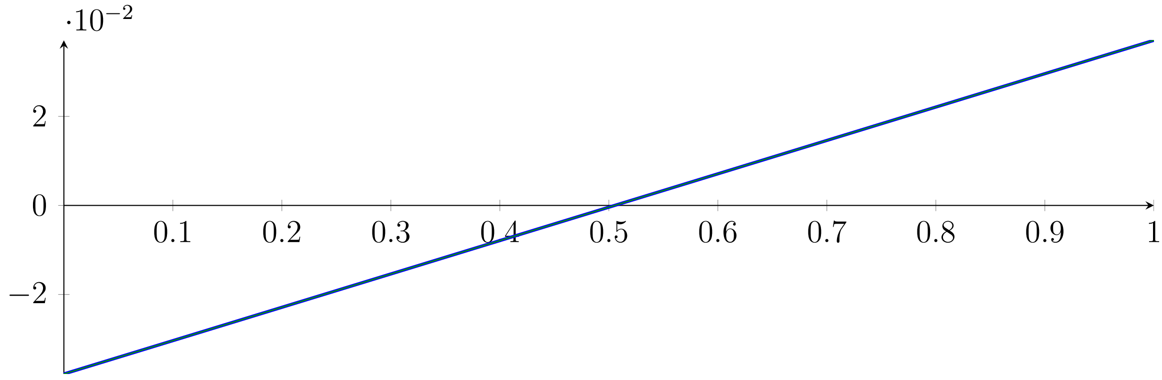
$$M = -0.00015083X^3 + 0.000286502X^2 + 0.0748131X - 0.0378581$$

$$m = -0.00015083X^3 + 0.000286502X^2 + 0.0748131X - 0.0378583$$

Root of M and m:

$$N(M) = \{-21.6009, 0.505318, 22.995\} \qquad N(m) = \{-21.6009, 0.50532, 22.995\}$$

Intersection intervals:



$[0.505318, 0.50532]$

Longest intersection interval: $2.33352 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.447832, 0.447832]$,

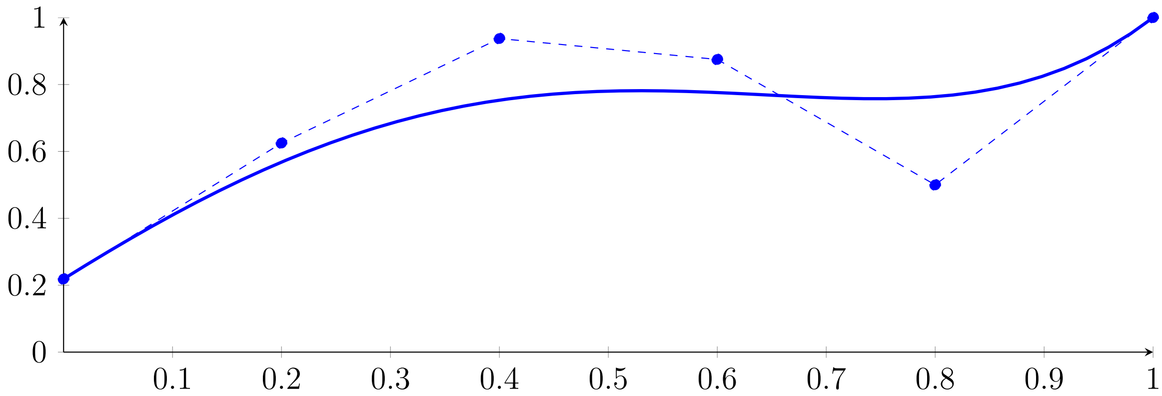
3.6 Recursion Branch 1 1 2 1 in Interval 1: $[0.447832, 0.447832]$

Found root in interval $[0.447832, 0.447832]$ at recursion depth 4!

3.7 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.78125X^5 + 1.71875X^4 - 2.8125X^3 - 0.9375X^2 + 2.03125X + 0.21875 \\ &= 0.21875B_{0,5}(X) + 0.625B_{1,5}(X) + 0.9375B_{2,5}(X) + 0.875B_{3,5}(X) + 0.5B_{4,5}(X) + 1B_{5,5}(X) \end{aligned}$$



Best approximation polynomials of degree 0, 1, 2, 3 and 4:

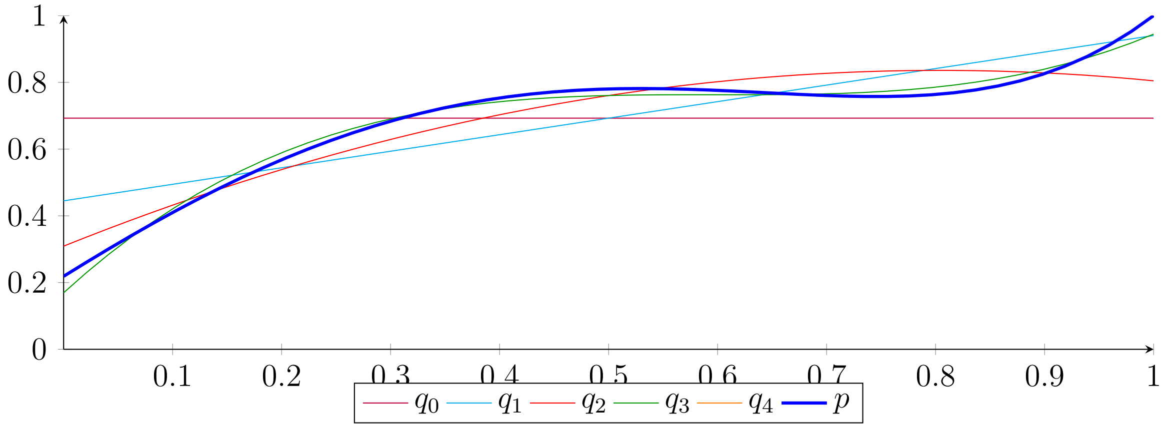
$$\begin{aligned} q_0 &= 0.692708 \\ &= 0.692708B_{0,0} \end{aligned}$$

$$\begin{aligned} q_1 &= 0.495536X + 0.44494 \\ &= 0.44494B_{0,1} + 0.940476B_{1,1} \end{aligned}$$

$$\begin{aligned} q_2 &= -0.814732X^2 + 1.31027X + 0.309152 \\ &= 0.309152B_{0,2} + 0.964286B_{1,2} + 0.804688B_{2,2} \end{aligned}$$

$$\begin{aligned} q_3 &= 2.79514X^3 - 5.00744X^2 + 2.98735X + 0.169395 \\ &= 0.169395B_{0,3} + 1.16518B_{1,3} + 0.491815B_{2,3} + 0.944444B_{3,3} \end{aligned}$$

$$\begin{aligned} q_4 &= 3.67187X^4 - 4.54861X^3 - 0.286458X^2 + 1.93824X + 0.22185 \\ &= 0.22185B_{0,4} + 0.706411B_{1,4} + 1.14323B_{2,4} + 0.395151B_{3,4} + 0.9969B_{4,4} \end{aligned}$$



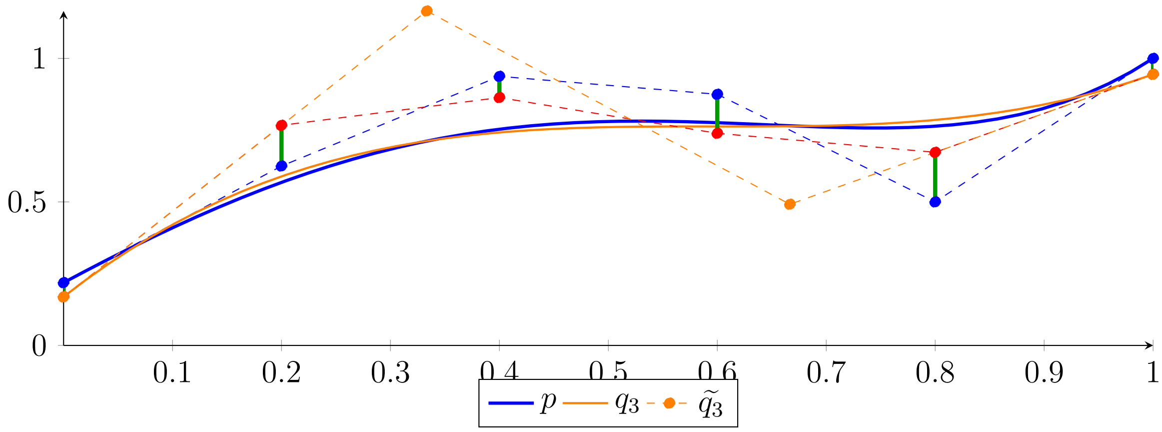
Degree reduction and raising matrices:

$$M_{5,3} = \begin{pmatrix} 0.960317 & -0.428571 & 0.166667 & -0.031746 \\ 0.126984 & 0.880952 & -0.428571 & 0.0873016 \\ -0.111111 & 0.761905 & 0.047619 & -0.031746 \\ -0.031746 & 0.047619 & 0.761905 & -0.111111 \\ 0.0873016 & -0.428571 & 0.880952 & 0.126984 \\ -0.031746 & 0.166667 & -0.428571 & 0.960317 \end{pmatrix} \quad M_{3,5} = \begin{pmatrix} 1 & 0.4 & \\ 1.97065\cdot 10^{-15} & 0.6 & \\ 3.08087\cdot 10^{-15} & -1.9984\cdot 10^{-15} & \\ -1.55431\cdot 10^{-15} & 1.77636\cdot 10^{-14} & -3.3 \end{pmatrix}$$

Degree reduction and raising:

$$\begin{aligned} q_3 &= 2.79514X^3 - 5.00744X^2 + 2.98735X + 0.169395 \\ &= 0.169395B_{0,3} + 1.16518B_{1,3} + 0.491815B_{2,3} + 0.944444B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -9.521\cdot 10^{-13}X^5 + 2.1004\cdot 10^{-12}X^4 + 2.79514X^3 - 5.00744X^2 + 2.98735X + 0.169395 \\ &= 0.169395B_{0,5} + 0.766865B_{1,5} + 0.863591B_{2,5} + 0.739087B_{3,5} + 0.672867B_{4,5} + 0.944444B_{5,5} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.172867$.
Bounding polynomials M and m:

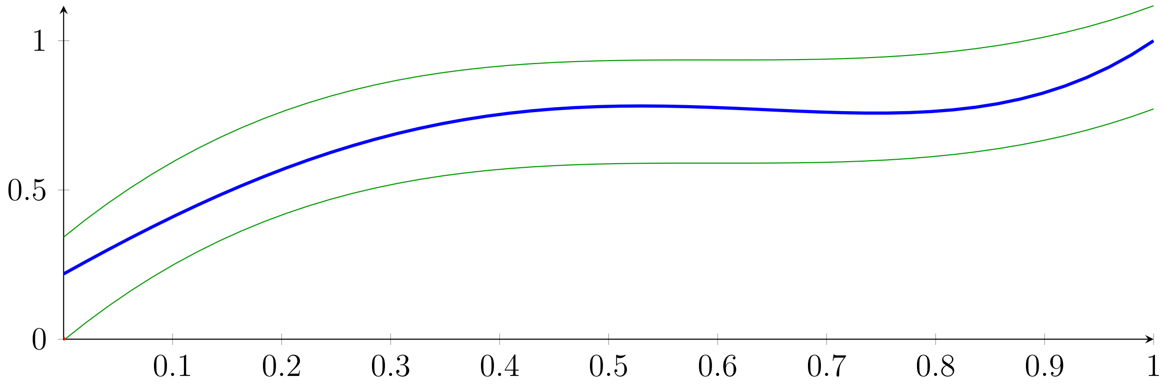
$$M = 2.79514X^3 - 5.00744X^2 + 2.98735X + 0.342262$$

$$m = 2.79514X^3 - 5.00744X^2 + 2.98735X - 0.00347222$$

Root of M and m:

$$N(M) = \{-0.0976984\} \qquad N(m) = \{0.00116458\}$$

Intersection intervals:



$[0, 0.00116458]$

Longest intersection interval: 0.00116458
$\Longrightarrow$ Selective recursion: interval 1: $[0.5, 0.500582]$,

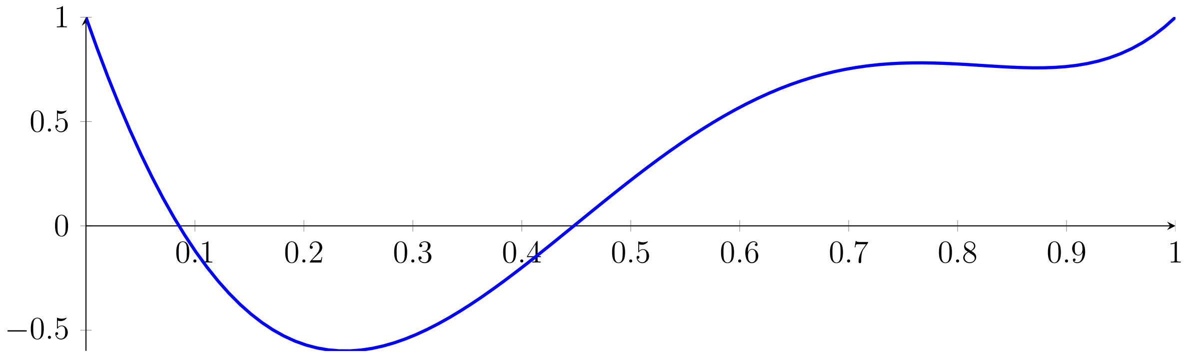
3.8 Recursion Branch 1 2 1 in Interval 1: $[0.5, 0.500582]$

Reached interval $[0.5, 0.500582]$ **without sign change** at depth 3!
p(0) = 0.21875 - p(1) 0.221114

3.9 Result: 2 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = 25X^5 - 35X^4 - 15X^3 + 40X^2 - 15X + 1$$



Result: Root Intervals

$$[0.0853575, 0.0853575],\ [0.447832, 0.447832]$$

with precision $\varepsilon = 0.001$.