

# Demo 2

This demo visualises the techniques of the algorithms by applying them to a polynomial with many different roots. For this purpose a polynomial of 8th degree is used, that has six different roots on the unit interval. The demo works with the standard datatype `long double` and precision  $\varepsilon = 0.001$ . Note that for this polynomial `BezClip` needs 29 recursive calls with maximum depth six, `QuadClip` needs 23 recursive calls with maximum depth five and `CubeClip` 21 recursive calls with maximum depth five, each time to isolate all six roots.

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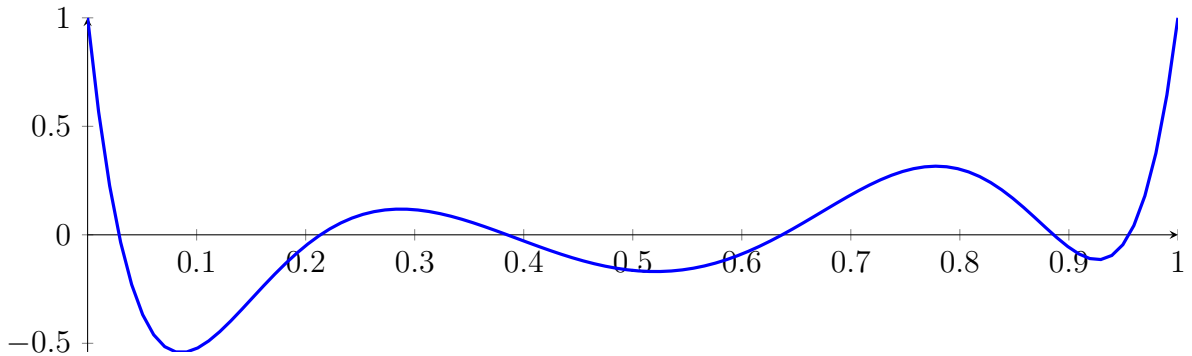
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# 1 BezClip Applied to a Polynomial of 8th Degree with Six Roots

$$2118X^8 - 8328X^7 + 14000X^6 - 13216X^5 + 7630X^4 - 2688X^3 + 532X^2 - 48X + 1$$

Called BezClip with input polynomial on interval  $[0, 1]$ :

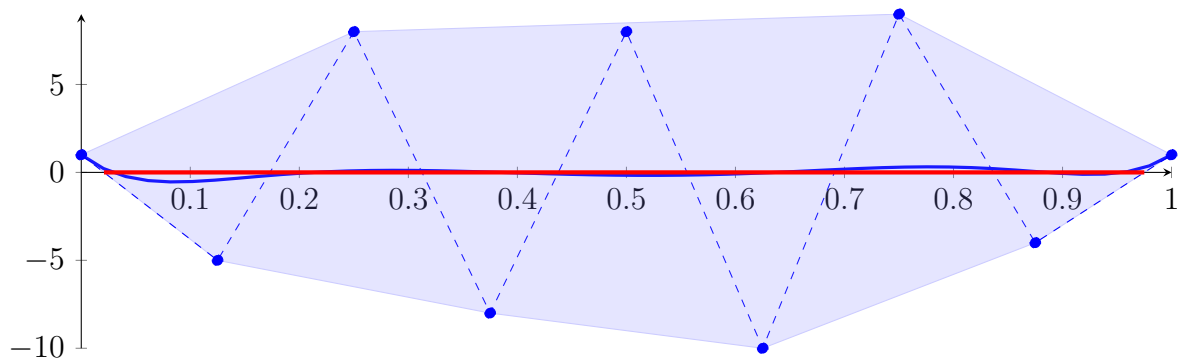
$$p = 2118X^8 - 8328X^7 + 14000X^6 - 13216X^5 + 7630X^4 - 2688X^3 + 532X^2 - 48X + 1$$



## 1.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 2118X^8 - 8328X^7 + 14000X^6 - 13216X^5 + 7630X^4 - 2688X^3 + 532X^2 - 48X + 1 \\ &= 1B_{0,8}(X) - 5B_{1,8}(X) + 8B_{2,8}(X) - 8B_{3,8}(X) + 8B_{4,8}(X) \\ &\quad - 10B_{5,8}(X) + 9B_{6,8}(X) - 4B_{7,8}(X) + 1B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.0208333, 0.975\}$$

Intersection intervals with the  $x$  axis:

$$[0.0208333, 0.975]$$

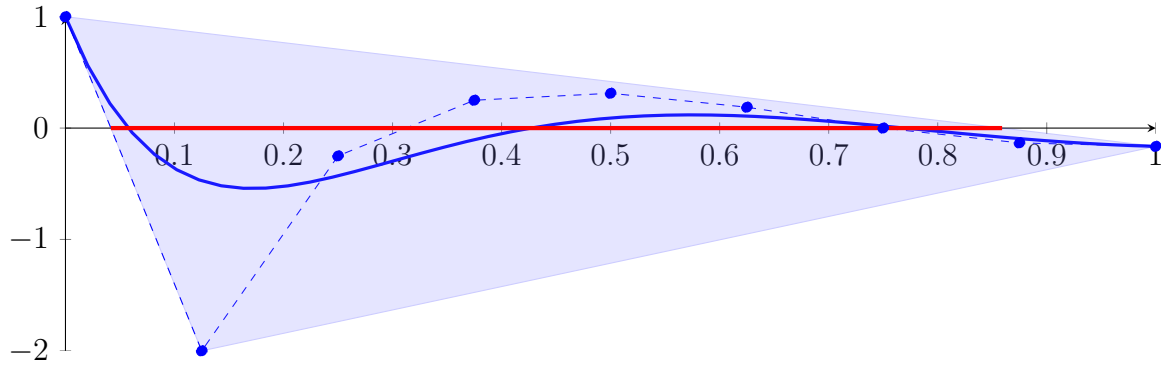
Longest intersection interval: 0.954167

$\implies$  Bisection: first half  $[0, 0.5]$  und second half  $[0.5, 1]$

## 1.2 Recursion Branch 1 1 on the First Half [0, 0.5]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 8.27344X^8 - 65.0625X^7 + 218.75X^6 - 413X^5 + 476.875X^4 - 336X^3 + 133X^2 - 24X + 1 \\
 &= 1B_{0,8}(X) - 2B_{1,8}(X) - 0.25B_{2,8}(X) + 0.25B_{3,8}(X) + 0.3125B_{4,8}(X) \\
 &\quad + 0.1875B_{5,8}(X) - 2.66443 \cdot 10^{-17}B_{6,8}(X) - 0.132813B_{7,8}(X) - 0.164063B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.0416667, 0.85906\}$$

Intersection intervals with the  $x$  axis:

$$[0.0416667, 0.85906]$$

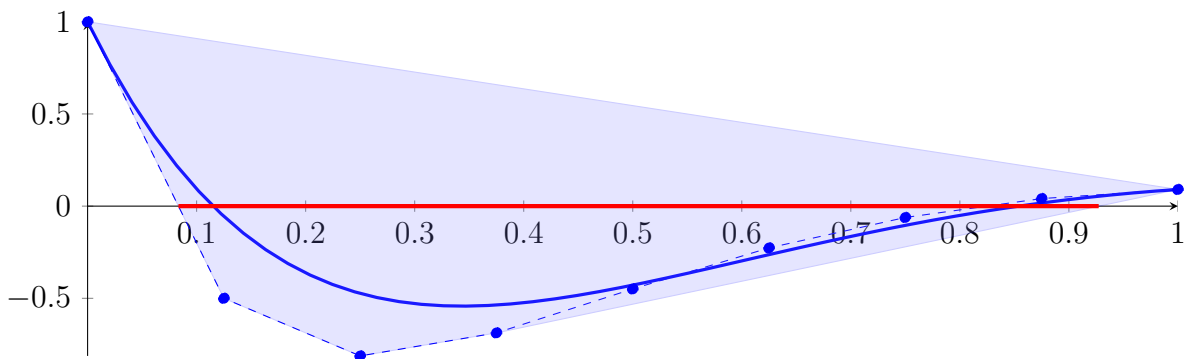
Longest intersection interval: 0.817394

⇒ Bisection: first half [0, 0.25] und second half [0.25, 0.5]

## 1.3 Recursion Branch 1 1 1 on the First Half [0, 0.25]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 0.0323181X^8 - 0.508301X^7 + 3.41797X^6 - 12.9063X^5 + 29.8047X^4 - 42X^3 + 33.25X^2 - 12X + 1 \\
 &= 1B_{0,8}(X) - 0.5B_{1,8}(X) - 0.8125B_{2,8}(X) - 0.6875B_{3,8}(X) - 0.449219B_{4,8}(X) \\
 &\quad - 0.226563B_{5,8}(X) - 0.0615234B_{6,8}(X) + 0.0409546B_{7,8}(X) + 0.0904236B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.0833333, 0.927352\}$$

Intersection intervals with the  $x$  axis:

$$[0.0833333, 0.927352]$$

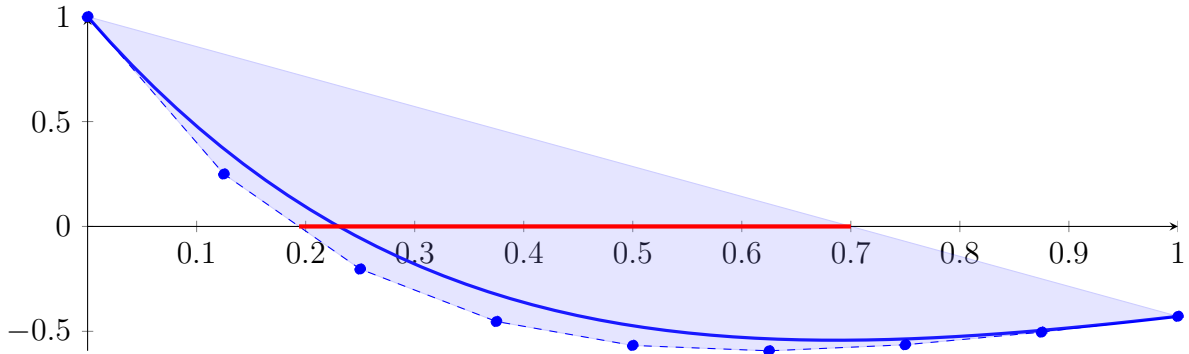
Longest intersection interval: 0.844018

⇒ Bisection: first half [0, 0.125] und second half [0.125, 0.25]

## 1.4 Recursion Branch 1 1 1 1 on the First Half [0, 0.125]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 0.000126243X^8 - 0.0039711X^7 + 0.0534058X^6 \\
 &\quad - 0.40332X^5 + 1.86279X^4 - 5.25X^3 + 8.3125X^2 - 6X + 1 \\
 &= 1B_{0,8}(X) + 0.25B_{1,8}(X) - 0.203125B_{2,8}(X) - 0.453125B_{3,8}(X) - 0.567139B_{4,8}(X) \\
 &\quad - 0.592896B_{5,8}(X) - 0.564011B_{6,8}(X) - 0.503869B_{7,8}(X) - 0.428466B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.193966, 0.700051\}$$

Intersection intervals with the  $x$  axis:

$$[0.193966, 0.700051]$$

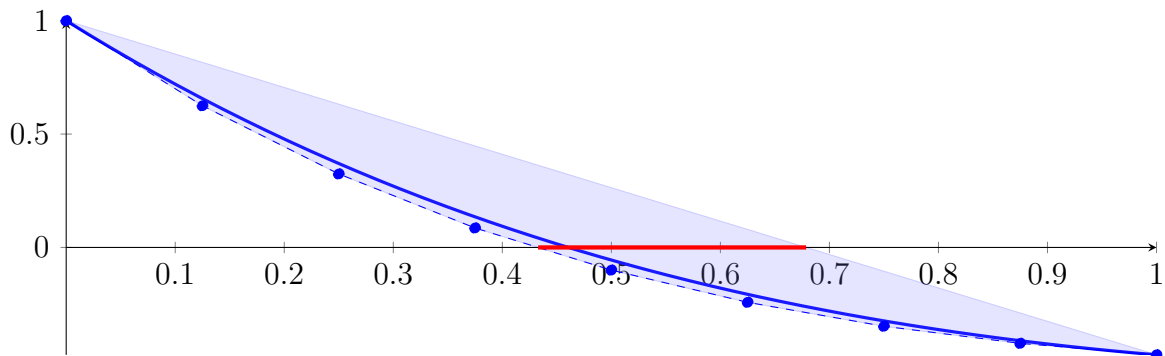
Longest intersection interval: 0.506086

$\implies$  Bisection: first half [0, 0.0625] und second half [0.0625, 0.125]

## 1.5 Recursion Branch 1 1 1 1 1 on the First Half [0, 0.0625]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 4.93135 \cdot 10^{-07}X^8 - 3.10242 \cdot 10^{-05}X^7 + 0.000834465X^6 \\
 &\quad - 0.0126038X^5 + 0.116425X^4 - 0.65625X^3 + 2.07812X^2 - 3X + 1 \\
 &= 1B_{0,8}(X) + 0.625B_{1,8}(X) + 0.324219B_{2,8}(X) + 0.0859375B_{3,8}(X) - 0.0998993B_{4,8}(X) \\
 &\quad - 0.241909B_{5,8}(X) - 0.347466B_{6,8}(X) - 0.422872B_{7,8}(X) - 0.4735B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.432804, 0.678656\}$$

Intersection intervals with the  $x$  axis:

$$[0.432804, 0.678656]$$

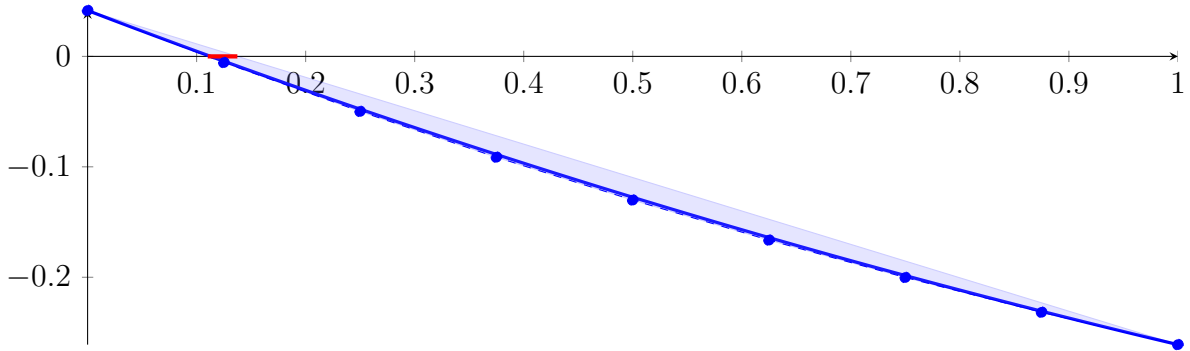
Longest intersection interval: 0.245852

$\implies$  Selective recursion: interval 1: [0.0270503, 0.042416],

## 1.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.0270503, 0.042416]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 6.58192 \cdot 10^{-12} X^8 - 1.59158 \cdot 10^{-09} X^7 + 1.64083 \cdot 10^{-07} X^6 - 9.48179 \cdot 10^{-06} X^5 \\
 &\quad + 0.000333945 X^4 - 0.00708805 X^3 + 0.0814231 X^2 - 0.377216 X + 0.0415556 \\
 &= 0.0415556 B_{0,8}(X) - 0.00559642 B_{1,8}(X) - 0.0498405 B_{2,8}(X) - 0.0913031 B_{3,8}(X) \\
 &\quad - 0.130106 B_{4,8}(X) - 0.166367 B_{5,8}(X) - 0.200198 B_{6,8}(X) - 0.231708 B_{7,8}(X) - 0.261001 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.110164, 0.137348\}$$

Intersection intervals with the  $x$  axis:

$$[0.110164, 0.137348]$$

Longest intersection interval: 0.0271844

⇒ Selective recursion: interval 1: [0.028743, 0.0291607],

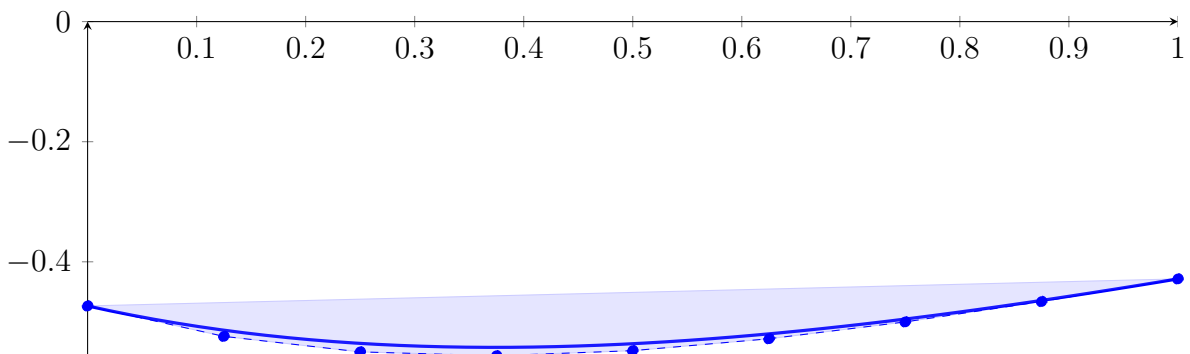
## 1.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: [0.028743, 0.0291607]

Found root in interval [0.028743, 0.0291607] at recursion depth 7!

## 1.8 Recursion Branch 1 1 1 1 2 on the Second Half [0.0625, 0.125]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 4.93135 \cdot 10^{-07} X^8 - 2.70791 \cdot 10^{-05} X^7 + 0.000631103 X^6 - 0.00822086 X^5 \\
 &\quad + 0.0648714 X^4 - 0.300958 X^3 + 0.693764 X^2 - 0.405027 X - 0.4735 \\
 &= -0.4735 B_{0,8}(X) - 0.524129 B_{1,8}(X) - 0.54998 B_{2,8}(X) - 0.556428 B_{3,8}(X) - 0.54792 B_{4,8}(X) \\
 &\quad - 0.528125 B_{5,8}(X) - 0.500054 B_{6,8}(X) - 0.466168 B_{7,8}(X) - 0.428466 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{\}$$

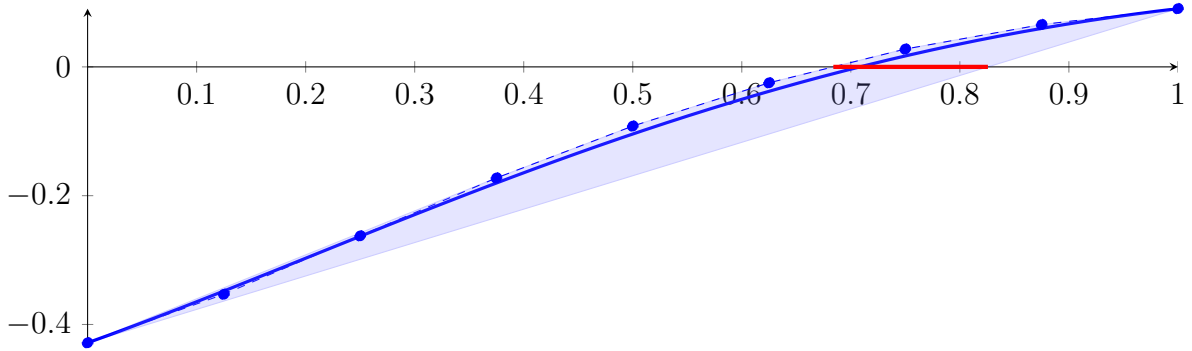
Intersection intervals with the  $x$  axis:

No intersection with the  $x$  axis. Done.

## 1.9 Recursion Branch 1 1 1 2 on the Second Half [0.125, 0.25]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 0.000126243X^8 - 0.00296116X^7 + 0.0291429X^6 - 0.159209X^5 \\
 &\quad + 0.517126X^4 - 0.895835X^3 + 0.427283X^2 + 0.603217X - 0.428466 \\
 &= -0.428466B_{0,8}(X) - 0.353064B_{1,8}(X) - 0.262402B_{2,8}(X) - 0.172477B_{3,8}(X) - 0.091898B_{4,8}(X) \\
 &\quad - 0.0247307B_{5,8}(X) + 0.0277023B_{6,8}(X) + 0.0656891B_{7,8}(X) + 0.0904236B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.683958, 0.825737\}$$

Intersection intervals with the  $x$  axis:

$$[0.683958, 0.825737]$$

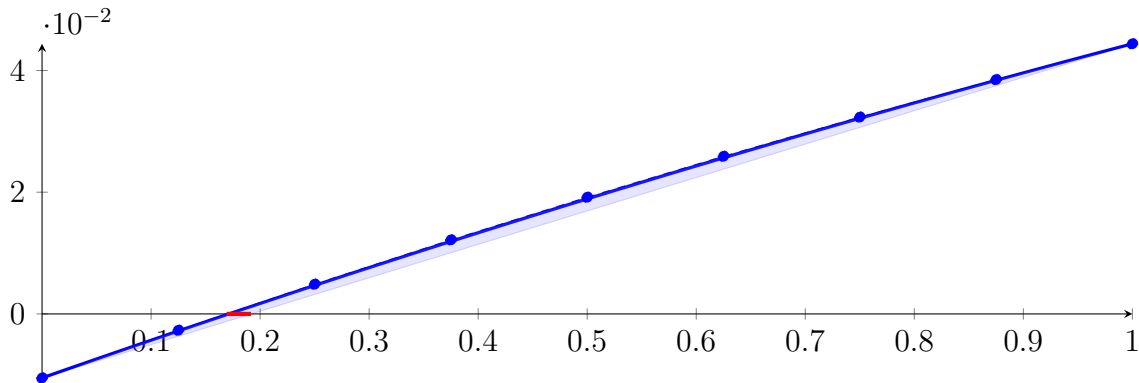
Longest intersection interval: 0.141779

$\implies$  Selective recursion: interval 1:  $[0.210495, 0.228217]$ ,

## 1.10 Recursion Branch 1 1 1 2 1 in Interval 1: [0.210495, 0.228217]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 2.06108 \cdot 10^{-11} X^8 - 2.61445 \cdot 10^{-09} X^7 + 1.34983 \cdot 10^{-07} X^6 - 3.80626 \cdot 10^{-06} X^5 \\
 &\quad + 5.89653 \cdot 10^{-05} X^4 - 0.00017378 X^3 - 0.00768011 X^2 + 0.0627 X - 0.0105174 \\
 &= -0.0105174B_{0,8}(X) - 0.00267989B_{1,8}(X) + 0.00488332B_{2,8}(X) \\
 &\quad + 0.0121691B_{3,8}(X) + 0.0191753B_{4,8}(X) + 0.0259003B_{5,8}(X) \\
 &\quad + 0.0323434B_{6,8}(X) + 0.0385045B_{7,8}(X) + 0.044384B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.169292, 0.191569\}$$

Intersection intervals with the  $x$  axis:

$$[0.169292, 0.191569]$$

Longest intersection interval: 0.0222772

$\implies$  Selective recursion: interval 1:  $[0.213495, 0.21389]$ ,

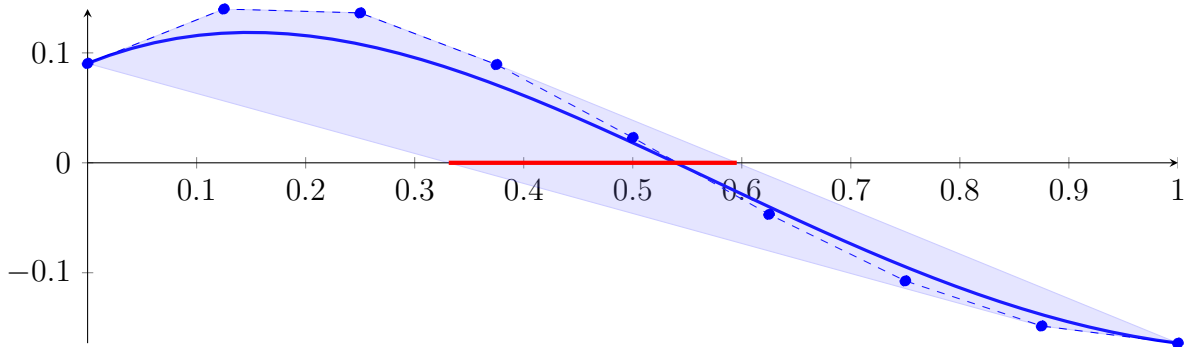
### 1.11 Recursion Branch 1 1 1 2 1 1 in Interval 1: [0.213495, 0.21389]

Found root in interval [0.213495, 0.21389] at recursion depth 6!

### 1.12 Recursion Branch 1 1 2 on the Second Half [0.25, 0.5]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 0.0323181X^8 - 0.249756X^7 + 0.764771X^6 - 1.26294X^5 \\
 &\quad + 1.01471X^4 + 0.534912X^3 - 1.48425X^2 + 0.395752X + 0.0904236 \\
 &= 0.0904236B_{0,8}(X) + 0.139893B_{1,8}(X) + 0.136353B_{2,8}(X) + 0.0893555B_{3,8}(X) + 0.0229492B_{4,8}(X) \\
 &\quad - 0.046875B_{5,8}(X) - 0.107422B_{6,8}(X) - 0.148438B_{7,8}(X) - 0.164063B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.331241, 0.595376\}$$

Intersection intervals with the  $x$  axis:

$$[0.331241, 0.595376]$$

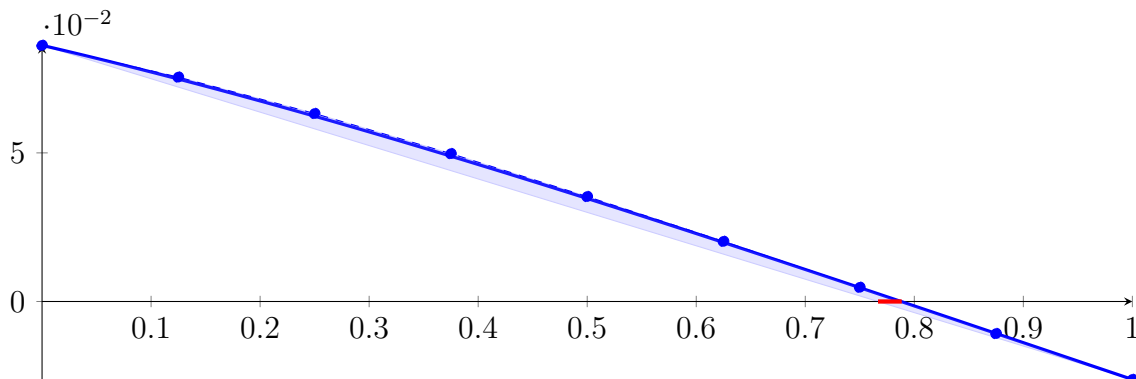
Longest intersection interval: 0.264135

⇒ Selective recursion: interval 1: [0.33281, 0.398844],

### 1.13 Recursion Branch 1 1 2 1 in Interval 1: [0.33281, 0.398844]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 7.6568 \cdot 10^{-07} X^8 - 1.47206 \cdot 10^{-05} X^7 + 9.67658 \cdot 10^{-05} X^6 - 0.000324878 X^5 \\
 &\quad - 0.00052946 X^4 + 0.0175348 X^3 - 0.0436261 X^2 - 0.0855454 X + 0.0861856 \\
 &= 0.0861856B_{0,8}(X) + 0.0754924B_{1,8}(X) + 0.0632411B_{2,8}(X) \\
 &\quad + 0.0497449B_{3,8}(X) + 0.0353093B_{4,8}(X) + 0.0202265B_{5,8}(X) \\
 &\quad + 0.00477301B_{6,8}(X) - 0.0107912B_{7,8}(X) - 0.0262226B_{8,8}(X)
 \end{aligned}$$





Intersection of the convex hull with the  $x$  axis:

$$\{0.76672, 0.788497\}$$

Intersection intervals with the  $x$  axis:

$$[0.76672, 0.788497]$$

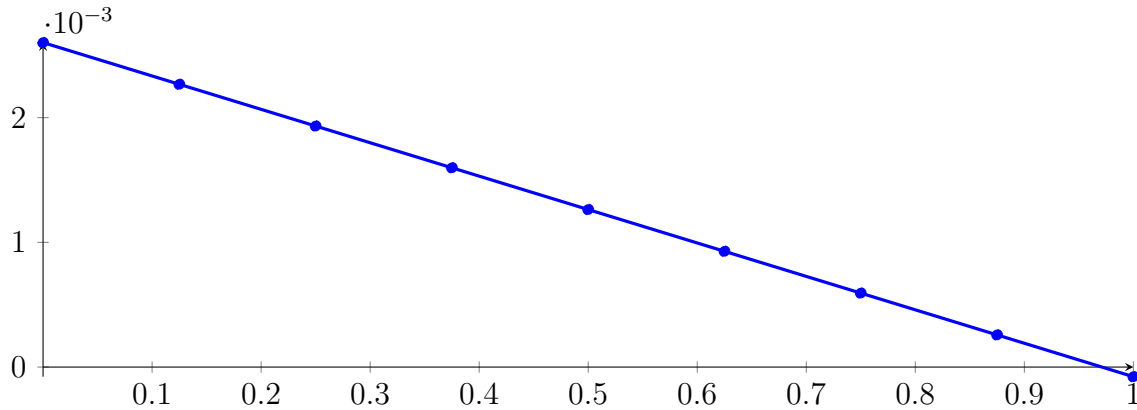
Longest intersection interval: 0.0217779

⇒ Selective recursion: interval 1: [0.38344, 0.384878],

### 1.14 Recursion Branch 1 1 2 1 1 in Interval 1: [0.38344, 0.384878]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.54807 \cdot 10^{-20} X^8 - 2.34357 \cdot 10^{-17} X^7 + 3.23863 \cdot 10^{-15} X^6 - 2.06354 \cdot 10^{-13} X^5 - 2.55381 \\ &\quad \cdot 10^{-10} X^4 + 1.51903 \cdot 10^{-07} X^3 - 2.94094 \cdot 10^{-06} X^2 - 0.00267653 X + 0.00260199 \\ &= 0.00260199 B_{0,8}(X) + 0.00226743 B_{1,8}(X) + 0.00193275 B_{2,8}(X) \\ &\quad + 0.00159798 B_{3,8}(X) + 0.00126311 B_{4,8}(X) + 0.000928136 B_{5,8}(X) \\ &\quad + 0.000593072 B_{6,8}(X) + 0.000257915 B_{7,8}(X) - 7.73294 \cdot 10^{-05} B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.971138, 0.971167\}$$

Intersection intervals with the  $x$  axis:

$$[0.971138, 0.971167]$$

Longest intersection interval:  $2.83875 \cdot 10^{-05}$

⇒ Selective recursion: interval 1: [0.384836, 0.384836],

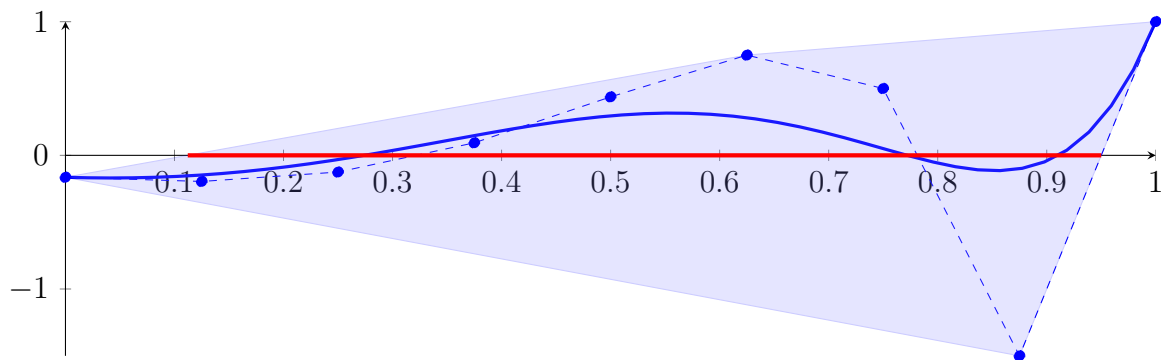
### 1.15 Recursion Branch 1 1 2 1 1 1 in Interval 1: [0.384836, 0.384836]

Found root in interval [0.384836, 0.384836] at recursion depth 6!

### 1.16 Recursion Branch 1 2 on the Second Half [0.5, 1]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 8.27344 X^8 + 1.125 X^7 - 5.03125 X^6 - 3.5 X^5 - 4.92188 X^4 + 2.625 X^3 + 2.84375 X^2 - 0.25 X - 0.164063 \\ &= -0.164063 B_{0,8}(X) - 0.195313 B_{1,8}(X) - 0.125 B_{2,8}(X) + 0.09375 B_{3,8}(X) \\ &\quad + 0.4375 B_{4,8}(X) + 0.75 B_{5,8}(X) + 0.5 B_{6,8}(X) - 1.5 B_{7,8}(X) + 1 B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.112179, 0.95\}$$

Intersection intervals with the  $x$  axis:

$$[0.112179, 0.95]$$

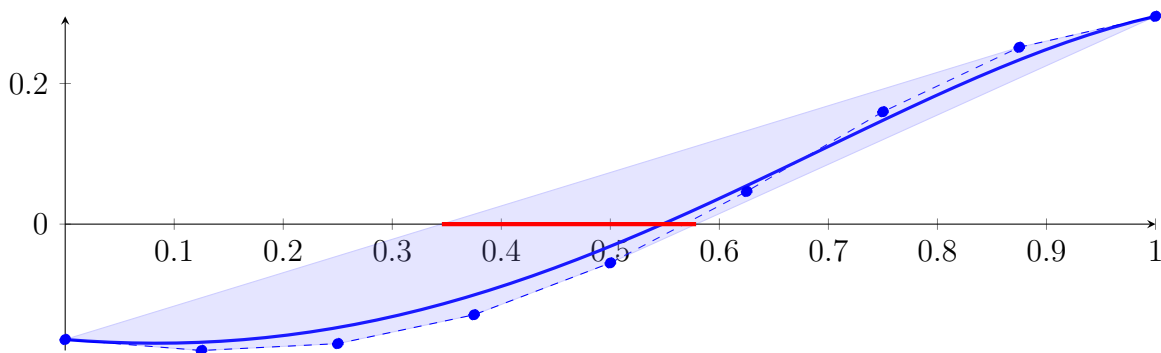
Longest intersection interval: 0.837821

⇒ Bisection: first half  $[0.5, 0.75]$  und second half  $[0.75, 1]$

### 1.17 Recursion Branch 1 2 1 on the First Half $[0.5, 0.75]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 0.0323181X^8 + 0.00878906X^7 - 0.0786133X^6 - 0.109375X^5 \\
 &\quad - 0.307617X^4 + 0.328125X^3 + 0.710937X^2 - 0.125X - 0.164063 \\
 &= -0.164063B_{0,8}(X) - 0.179688B_{1,8}(X) - 0.169922B_{2,8}(X) - 0.128906B_{3,8}(X) - 0.0551758B_{4,8}(X) \\
 &\quad + 0.0463867B_{5,8}(X) + 0.15979B_{6,8}(X) + 0.251465B_{7,8}(X) + 0.295502B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.345476, 0.57867\}$$

Intersection intervals with the  $x$  axis:

$$[0.345476, 0.57867]$$

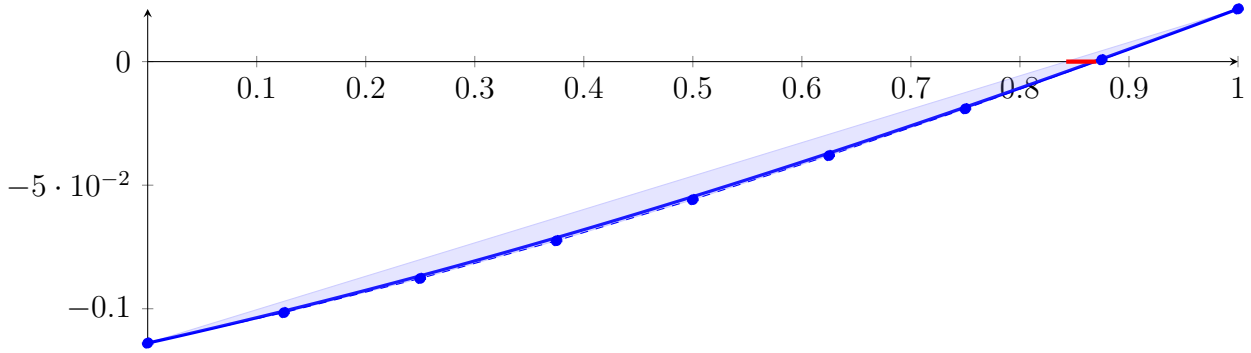
Longest intersection interval: 0.233194

⇒ Selective recursion: interval 1:  $[0.586369, 0.644668]$ ,

### 1.18 Recursion Branch 1 2 1 1 in Interval 1: [0.586369, 0.644668]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 2.82612 \cdot 10^{-07} X^8 + 3.67909 \cdot 10^{-06} X^7 + 8.14422 \cdot 10^{-06} X^6 - 0.000121143 X^5 \\
 &\quad - 0.00175175 X^4 - 0.00353873 X^3 + 0.0419415 X^2 + 0.0986692 X - 0.113907 \\
 &= -0.113907 B_{0,8}(X) - 0.101573 B_{1,8}(X) - 0.0877413 B_{2,8}(X) - 0.072475 B_{3,8}(X) - 0.0558622 B_{4,8}(X) \\
 &\quad - 0.0380183 B_{5,8}(X) - 0.0190878 B_{6,8}(X) + 0.00075471 B_{7,8}(X) + 0.0213047 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.842434, 0.870246\}$$

Intersection intervals with the  $x$  axis:

$$[0.842434, 0.870246]$$

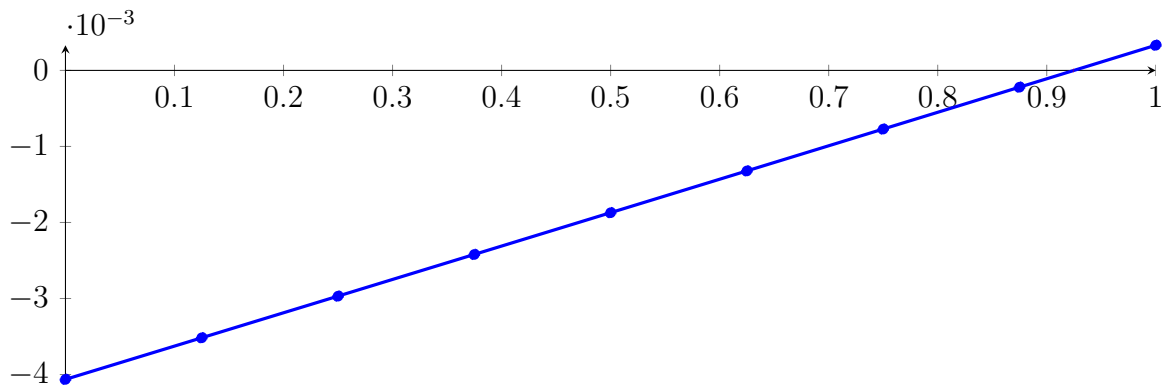
Longest intersection interval: 0.027812

⇒ Selective recursion: interval 1: [0.635482, 0.637103],

### 1.19 Recursion Branch 1 2 1 1 1 in Interval 1: [0.635482, 0.637103]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 2.34205 \cdot 10^{-19} X^8 + 7.20961 \cdot 10^{-17} X^7 + 1.64098 \cdot 10^{-14} X^6 - 2.60981 \cdot 10^{-13} X^5 - 1.2495 \\
 &\quad \cdot 10^{-09} X^4 - 2.17977 \cdot 10^{-07} X^3 + 1.92694 \cdot 10^{-05} X^2 + 0.00437586 X - 0.00406386 \\
 &= -0.00406386 B_{0,8}(X) - 0.00351688 B_{1,8}(X) - 0.00296921 B_{2,8}(X) \\
 &\quad - 0.00242085 B_{3,8}(X) - 0.00187182 B_{4,8}(X) - 0.00132211 B_{5,8}(X) \\
 &\quad - 0.000771723 B_{6,8}(X) - 0.00022067 B_{7,8}(X) + 0.000331048 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.924675, 0.924996\}$$

Intersection intervals with the  $x$  axis:

$$[0.924675, 0.924996]$$

Longest intersection interval: 0.000321349

⇒ Selective recursion: interval 1: [0.636981, 0.636981],

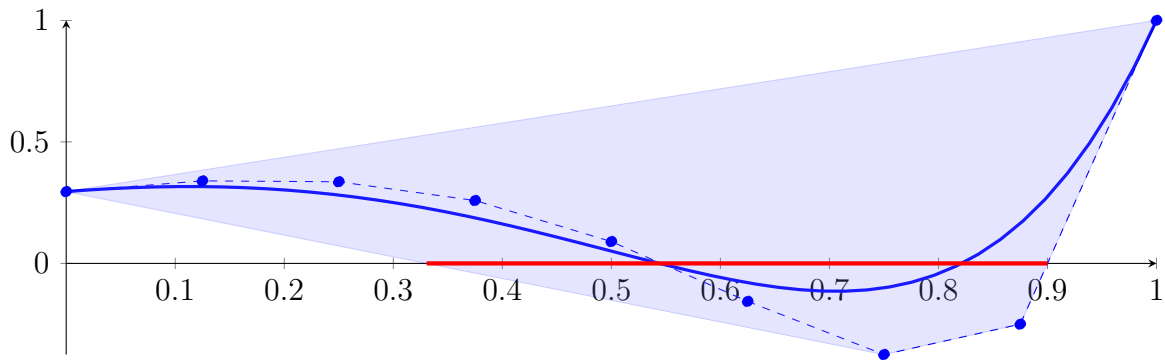
## 1.20 Recursion Branch 1 2 1 1 1 1 in Interval 1: [0.636981, 0.636981]

Found root in interval [0.636981, 0.636981] at recursion depth 6!

## 1.21 Recursion Branch 1 2 2 on the Second Half [0.75, 1]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 0.0323181X^8 + 0.267334X^7 + 0.887817X^6 + 1.41333X^5 \\
 &\quad + 0.536194X^4 - 1.45093X^3 - 1.33386X^2 + 0.352295X + 0.295502 \\
 &= 0.295502B_{0,8}(X) + 0.339539B_{1,8}(X) + 0.335937B_{2,8}(X) + 0.258789B_{3,8}(X) \\
 &\quad + 0.0898437B_{4,8}(X) - 0.15625B_{5,8}(X) - 0.375B_{6,8}(X) - 0.25B_{7,8}(X) + 1B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.330538, 0.9\}$$

Intersection intervals with the  $x$  axis:

$$[0.330538, 0.9]$$

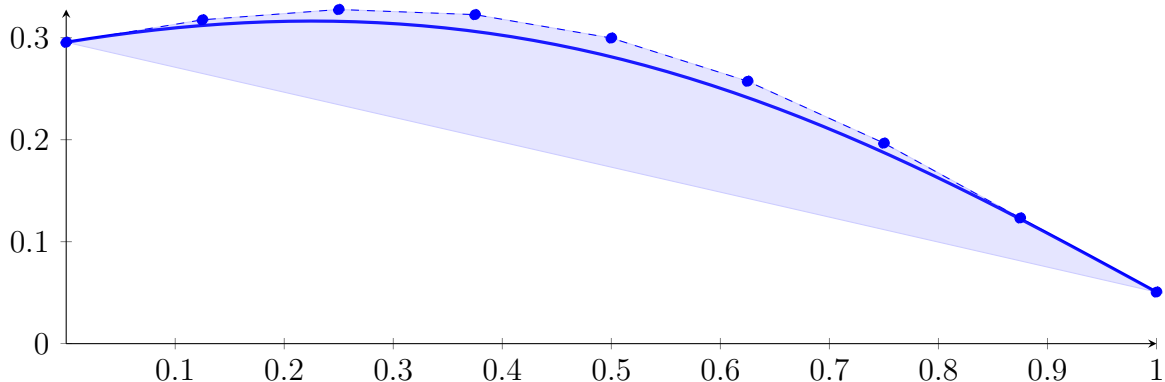
Longest intersection interval: 0.569462

⇒ Bisection: first half [0.75, 0.875] und second half [0.875, 1]

## 1.22 Recursion Branch 1 2 2 1 on the First Half [0.75, 0.875]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 0.000126243X^8 + 0.00208855X^7 + 0.0138721X^6 + 0.0441666X^5 \\
 &\quad + 0.0335121X^4 - 0.181366X^3 - 0.333466X^2 + 0.176147X + 0.295502 \\
 &= 0.295502B_{0,8}(X) + 0.31752B_{1,8}(X) + 0.327629B_{2,8}(X) + 0.32259B_{3,8}(X) + 0.299643B_{4,8}(X) \\
 &\quad + 0.257295B_{5,8}(X) + 0.196605B_{6,8}(X) + 0.123225B_{7,8}(X) + 0.0505832B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{\}$$

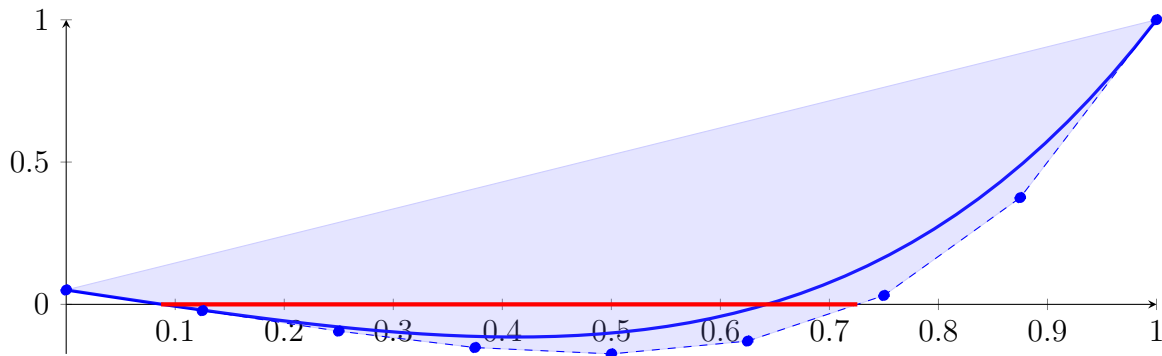
Intersection intervals with the  $x$  axis:

No intersection with the  $x$  axis. Done.

### 1.23 Recursion Branch 1 2 2 2 on the Second Half $[0.875, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 0.000126243X^8 + 0.00309849X^7 + 0.0320268X^6 + 0.178329X^5 \\ &\quad + 0.544363X^4 + 0.75196X^3 + 0.0206513X^2 - 0.581138X + 0.0505832 \\ &= 0.0505832B_{0,8}(X) - 0.022059B_{1,8}(X) - 0.0939636B_{2,8}(X) - 0.151703B_{3,8}(X) \\ &\quad - 0.174072B_{4,8}(X) - 0.128906B_{5,8}(X) + 0.03125B_{6,8}(X) + 0.375B_{7,8}(X) + 1B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.0870418, 0.72561\}$$

Intersection intervals with the  $x$  axis:

$$[0.0870418, 0.72561]$$

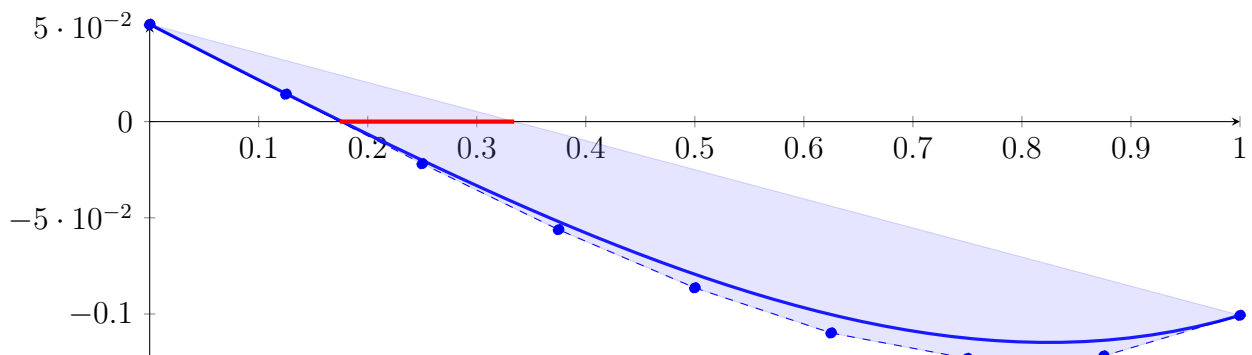
Longest intersection interval: 0.638568

⇒ Bisection: first half  $[0.875, 0.9375]$  und second half  $[0.9375, 1]$

### 1.24 Recursion Branch 1 2 2 2 1 on the First Half $[0.875, 0.9375]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 4.93135 \cdot 10^{-07} X^8 + 2.42069 \cdot 10^{-05} X^7 + 0.000500418X^6 + 0.00557277X^5 \\ &\quad + 0.0340227X^4 + 0.093995X^3 + 0.00516284X^2 - 0.290569X + 0.0505832 \\ &= 0.0505832B_{0,8}(X) + 0.0142621B_{1,8}(X) - 0.0218746B_{2,8}(X) - 0.0561484B_{3,8}(X) \\ &\quad - 0.0863949B_{4,8}(X) - 0.109864B_{5,8}(X) - 0.123102B_{6,8}(X) - 0.121816B_{7,8}(X) - 0.100707B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.174334, 0.334345\}$$

Intersection intervals with the  $x$  axis:

$$[0.174334, 0.334345]$$

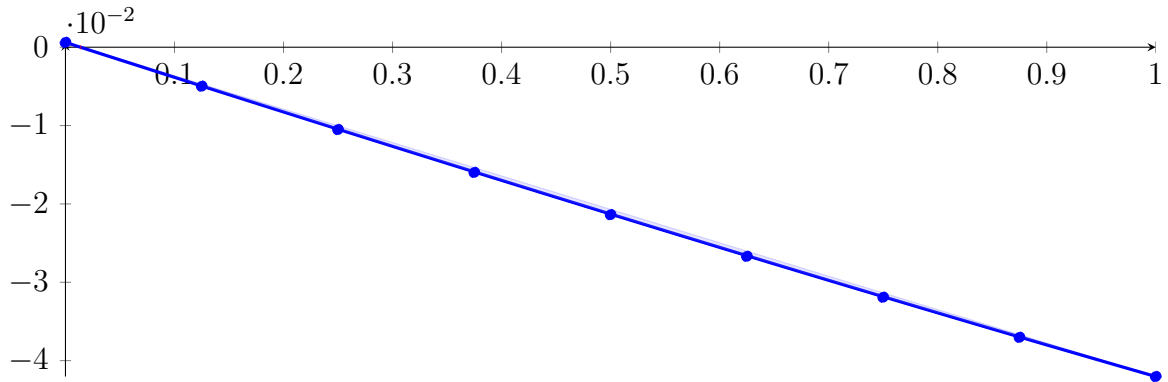
Longest intersection interval: 0.160011

⇒ Selective recursion: [interval 1: \[0.885896, 0.895897\]](#),

## 1.25 Recursion Branch 1 2 2 2 1 1 in Interval 1: [0.885896, 0.895897]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 2.11922 \cdot 10^{-13} X^8 + 6.68593 \cdot 10^{-11} X^7 + 8.90206 \cdot 10^{-09} X^6 + 6.41096 \cdot 10^{-07} X^5 \\ &\quad + 2.56403 \cdot 10^{-05} X^4 + 0.000489444 X^3 + 0.00155744 X^2 - 0.0447154 X + 0.000614501 \\ &= 0.000614501 B_{0,8}(X) - 0.00497492 B_{1,8}(X) - 0.0105087 B_{2,8}(X) \\ &\quad - 0.0159782 B_{3,8}(X) - 0.0213741 B_{4,8}(X) - 0.0266871 B_{5,8}(X) \\ &\quad - 0.0319073 B_{6,8}(X) - 0.0370244 B_{7,8}(X) - 0.0420277 B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.0137425, 0.0144106\}$$

Intersection intervals with the  $x$  axis:

$$[0.0137425, 0.0144106]$$

Longest intersection interval: 0.000668131

⇒ Selective recursion: [interval 1: \[0.886033, 0.88604\]](#),

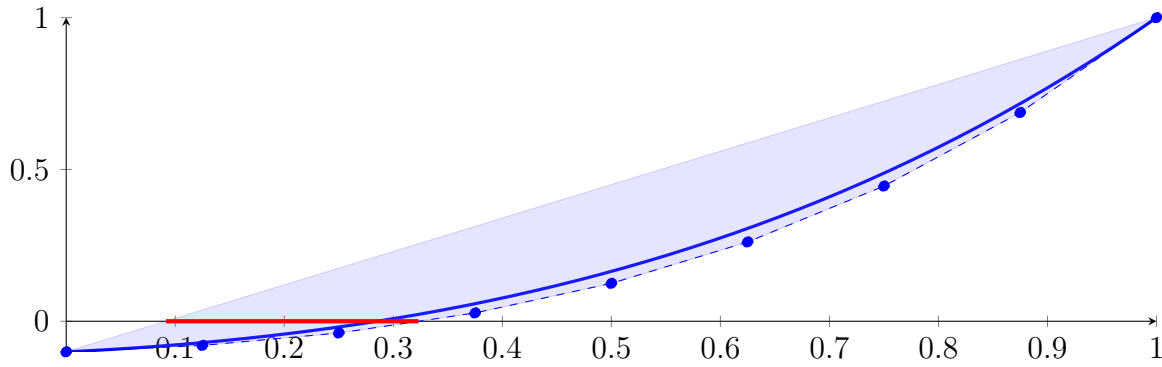
## 1.26 Recursion Branch 1 2 2 2 1 1 1 in Interval 1: [0.886033, 0.88604]

Found root in interval [0.886033, 0.88604] at recursion depth 7!

## 1.27 Recursion Branch 1 2 2 2 2 on the Second Half [0.9375, 1]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 4.93135 \cdot 10^{-07} X^8 + 2.8152 \cdot 10^{-05} X^7 + 0.000683675 X^6 + 0.00911124 X^5 \\ &\quad + 0.0702746 X^4 + 0.296697 X^3 + 0.55504 X^2 + 0.168872 X - 0.100707 \\ &= -0.100707 B_{0,8}(X) - 0.0795981 B_{1,8}(X) - 0.0386662 B_{2,8}(X) + 0.0273867 B_{3,8}(X) \\ &\quad + 0.124863 B_{4,8}(X) + 0.26123 B_{5,8}(X) + 0.445312 B_{6,8}(X) + 0.6875 B_{7,8}(X) + 1 B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.0914932, 0.323173\}$$

Intersection intervals with the  $x$  axis:

$$[0.0914932, 0.323173]$$

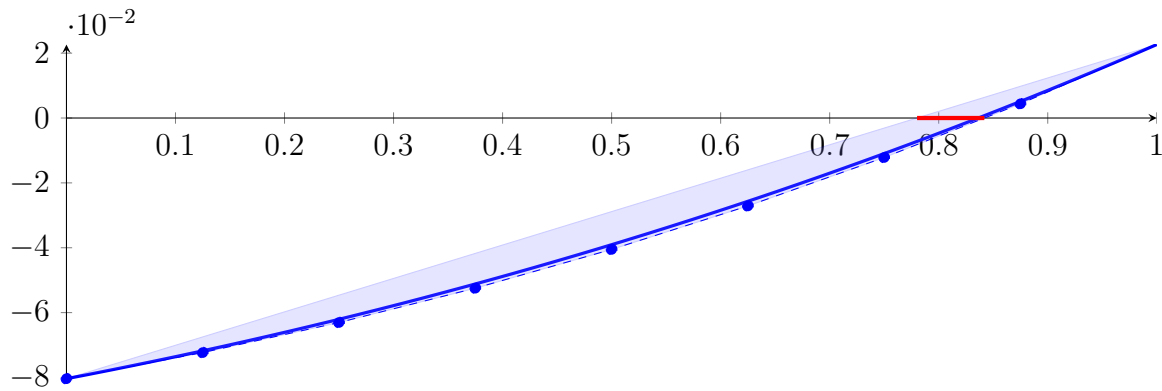
Longest intersection interval: 0.23168

$\implies$  Selective recursion: interval 1:  $[0.943218, 0.957698]$ ,

## 1.28 Recursion Branch 1 2 2 2 2 1 in Interval 1: $[0.943218, 0.957698]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 4.09327 \cdot 10^{-12} X^8 + 1.02155 \cdot 10^{-09} X^7 + 1.08531 \cdot 10^{-07} X^6 + 6.33542 \cdot 10^{-06} X^5 \\
 &\quad + 0.000214723 X^4 + 0.00401902 X^3 + 0.0343565 X^2 + 0.0644316 X - 0.0803781 \\
 &= -0.0803781 B_{0,8}(X) - 0.0723241 B_{1,8}(X) - 0.0630431 B_{2,8}(X) - 0.0524634 B_{3,8}(X) \\
 &\quad - 0.04051 B_{4,8}(X) - 0.027105 B_{5,8}(X) - 0.012167 B_{6,8}(X) + 0.00438861 B_{7,8}(X) + 0.0226502 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the  $x$  axis:

$$\{0.780155, 0.841865\}$$

Intersection intervals with the  $x$  axis:

$$[0.780155, 0.841865]$$

Longest intersection interval: 0.0617093

$\implies$  Selective recursion: interval 1:  $[0.954515, 0.955409]$ ,

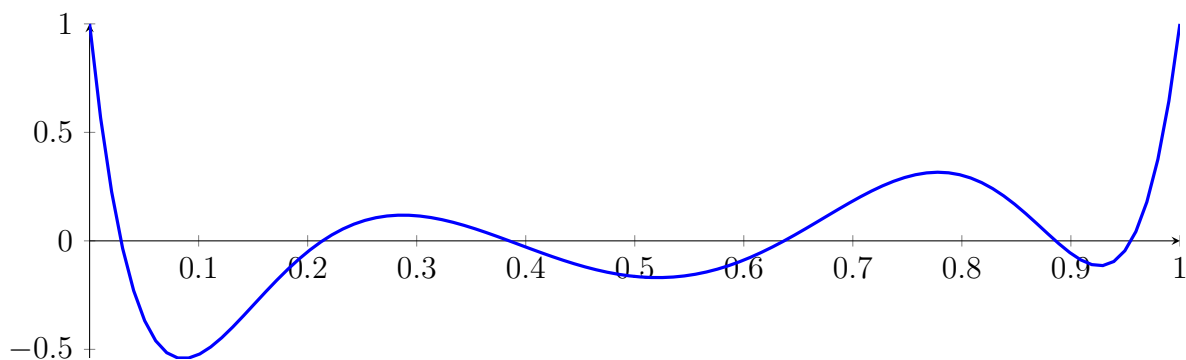
## 1.29 Recursion Branch 1 2 2 2 2 1 1 in Interval 1: $[0.954515, 0.955409]$

Found root in interval  $[0.954515, 0.955409]$  at recursion depth 7!

### 1.30 Result: 6 Root Intervals

Input Polynomial on Interval  $[0, 1]$

$$p = 2118X^8 - 8328X^7 + 14000X^6 - 13216X^5 + 7630X^4 - 2688X^3 + 532X^2 - 48X + 1$$



**Result: Root Intervals**

$$[0.028743, 0.0291607], [0.213495, 0.21389], [0.384836, 0.384836], [0.636981, 0.636981], \\ [0.886033, 0.88604], [0.954515, 0.955409]$$

with precision  $\varepsilon = 0.001$ .

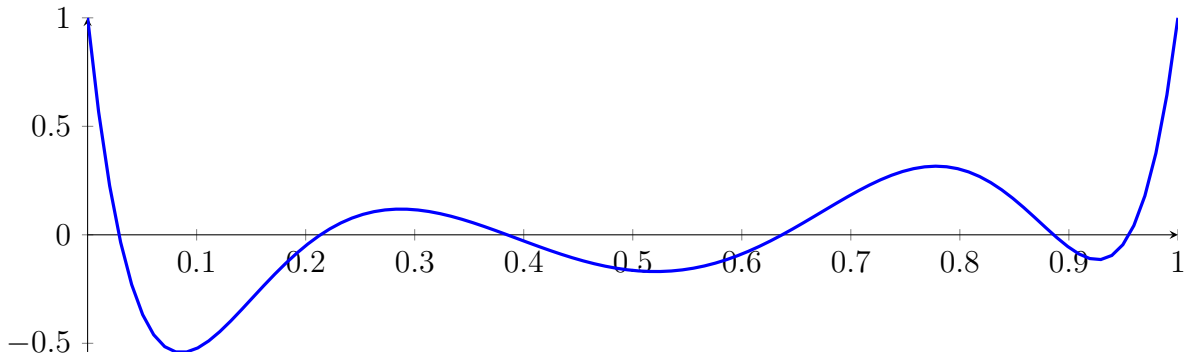


## 2 QuadClip Applied to a Polynomial of 8th Degree with Six Roots

$$2118X^8 - 8328X^7 + 14000X^6 - 13216X^5 + 7630X^4 - 2688X^3 + 532X^2 - 48X + 1$$

Called QuadClip with input polynomial on interval  $[0, 1]$ :

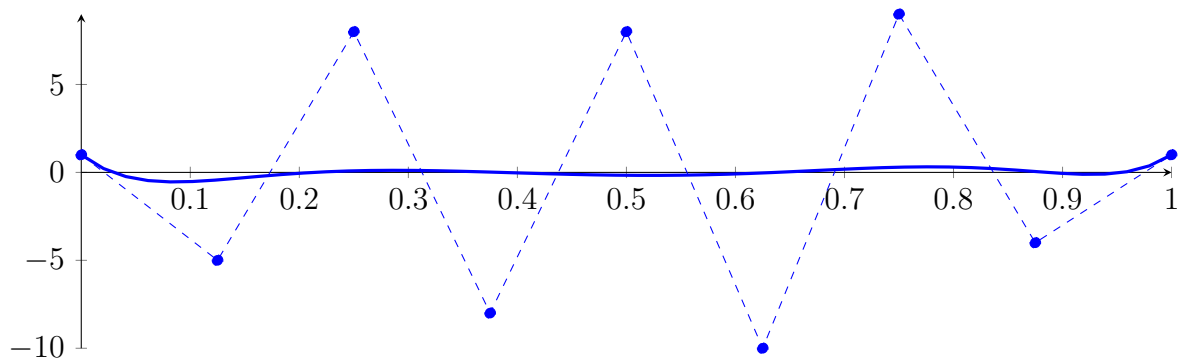
$$p = 2118X^8 - 8328X^7 + 14000X^6 - 13216X^5 + 7630X^4 - 2688X^3 + 532X^2 - 48X + 1$$



### 2.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

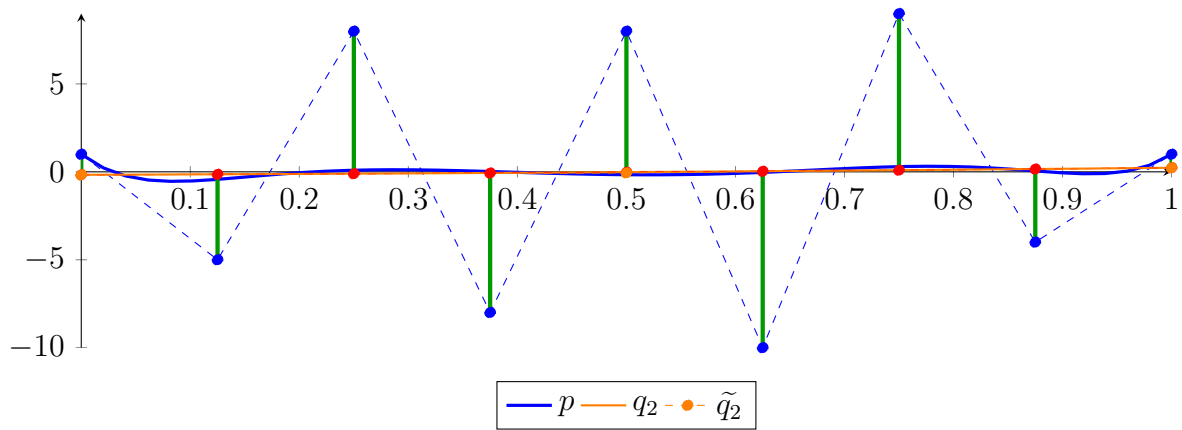
$$\begin{aligned} p &= 2118X^8 - 8328X^7 + 14000X^6 - 13216X^5 + 7630X^4 - 2688X^3 + 532X^2 - 48X + 1 \\ &= 1B_{0,8}(X) - 5B_{1,8}(X) + 8B_{2,8}(X) - 8B_{3,8}(X) + 8B_{4,8}(X) \\ &\quad - 10B_{5,8}(X) + 9B_{6,8}(X) - 4B_{7,8}(X) + 1B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.181818X^2 + 0.218182X - 0.169697 \\ &= -0.169697B_{0,2} - 0.0606061B_{1,2} + 0.230303B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -5.17678 \cdot 10^{-14}X^8 + 1.8174 \cdot 10^{-13}X^7 - 2.51464 \cdot 10^{-13}X^6 + 1.74232 \cdot 10^{-13}X^5 \\ &\quad - 6.35016 \cdot 10^{-14}X^4 + 1.1649 \cdot 10^{-14}X^3 + 0.181818X^2 + 0.218182X - 0.169697 \\ &= -0.169697B_{0,8} - 0.142424B_{1,8} - 0.108658B_{2,8} - 0.0683983B_{3,8} - 0.021645B_{4,8} \\ &\quad + 0.0316017B_{5,8} + 0.091342B_{6,8} + 0.157576B_{7,8} + 0.230303B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 10.0316$ .

**Bounding polynomials  $M$  and  $m$ :**

$$M = 0.181818X^2 + 0.218182X + 9.8619$$

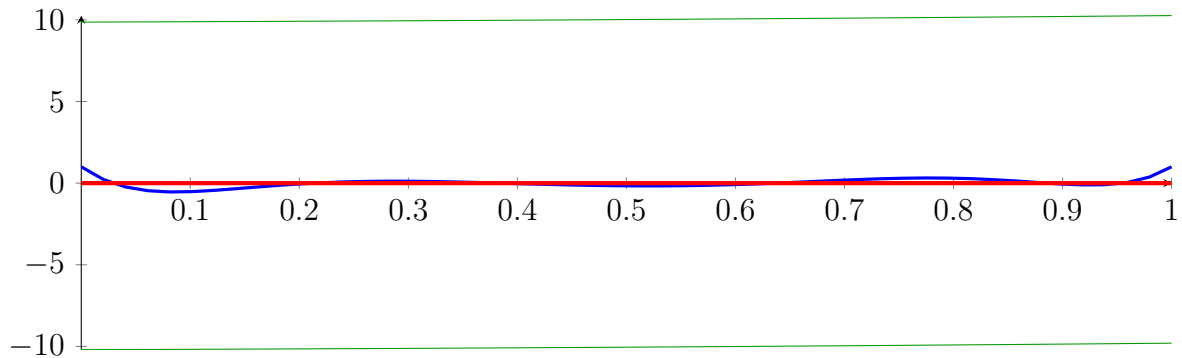
$$m = 0.181818X^2 + 0.218182X - 10.2013$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{\}$$

$$N(m) = \{-8.11446, 6.91446\}$$

**Intersection intervals:**



$$[0, 1]$$

Longest intersection interval: 1

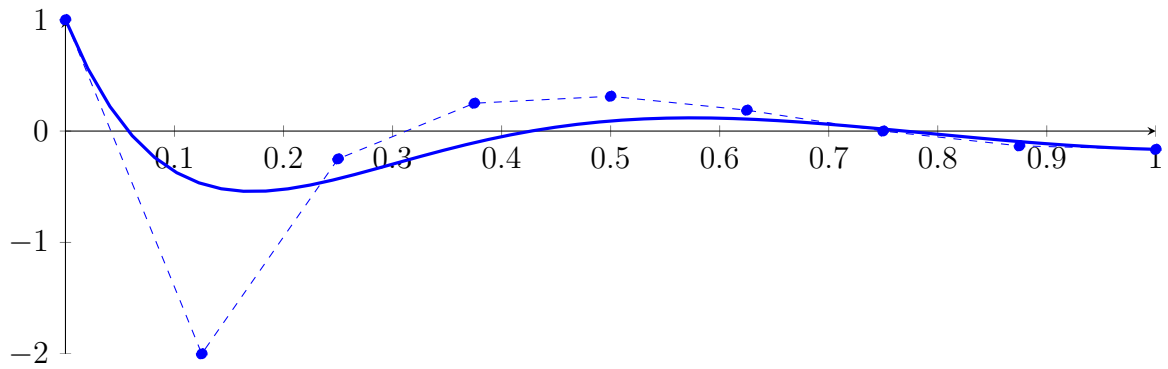
$\implies$  Bisection: first half  $[0, 0.5]$  und second half  $[0.5, 1]$

Bisection point is very near to a root?!?

## 2.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

**Normalized monomial und Bézier representations and the Bézier polygon:**

$$\begin{aligned} p &= 8.27344X^8 - 65.0625X^7 + 218.75X^6 - 413X^5 + 476.875X^4 - 336X^3 + 133X^2 - 24X + 1 \\ &= 1B_{0,8}(X) - 2B_{1,8}(X) - 0.25B_{2,8}(X) + 0.25B_{3,8}(X) + 0.3125B_{4,8}(X) \\ &\quad + 0.1875B_{5,8}(X) - 2.66443 \cdot 10^{-17}B_{6,8}(X) - 0.132813B_{7,8}(X) - 0.164063B_{8,8}(X) \end{aligned}$$



**Degree reduction and raising:**

$$q_2 = -0.194602X^2 + 0.378977X - 0.213163$$

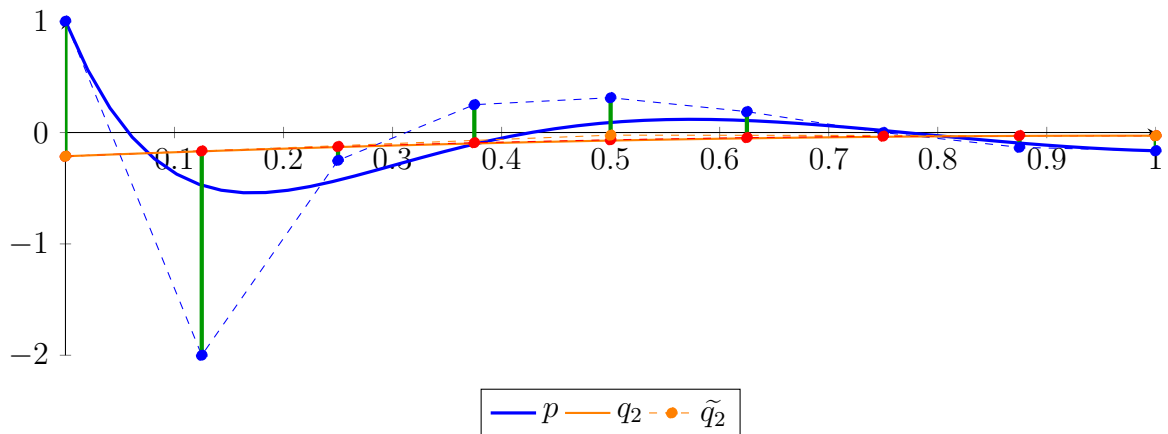
$$= -0.213163B_{0,2} - 0.0236742B_{1,2} - 0.0287879B_{2,2}$$

$$\tilde{q}_2 = 1.98551 \cdot 10^{-13}X^8 - 8.06618 \cdot 10^{-13}X^7 + 1.34139 \cdot 10^{-12}X^6 - 1.17158 \cdot 10^{-12}X^5$$

$$+ 5.71606 \cdot 10^{-13}X^4 - 1.51872 \cdot 10^{-13}X^3 - 0.194602X^2 + 0.378977X - 0.213163$$

$$= -0.213163B_{0,8} - 0.165791B_{1,8} - 0.125369B_{2,8} - 0.0918966B_{3,8} - 0.0653747B_{4,8}$$

$$- 0.0458029B_{5,8} - 0.0331811B_{6,8} - 0.0275095B_{7,8} - 0.0287879B_{8,8}$$



The maximum difference of the Bézier coefficients is  $\delta = 1.83421$ .

**Bounding polynomials  $M$  and  $m$ :**

$$M = -0.194602X^2 + 0.378977X + 1.62105$$

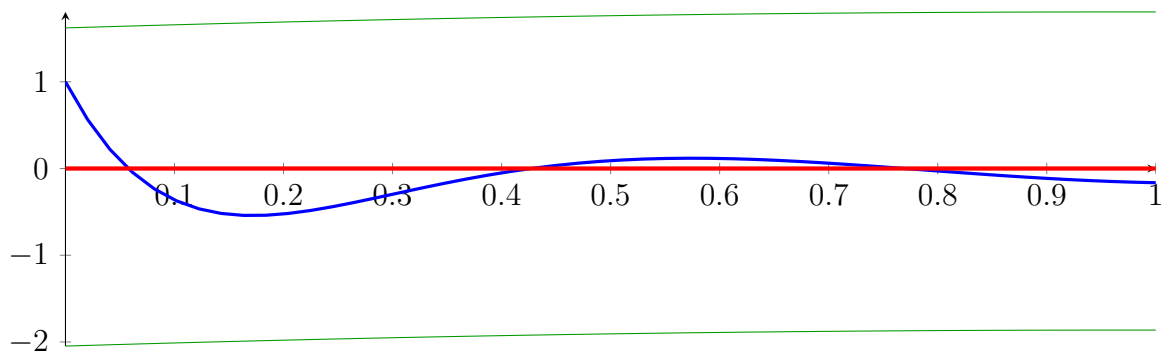
$$m = -0.194602X^2 + 0.378977X - 2.04737$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{-2.07229, 4.01973\}$$

$$N(m) = \{\}$$

**Intersection intervals:**



[0, 1]

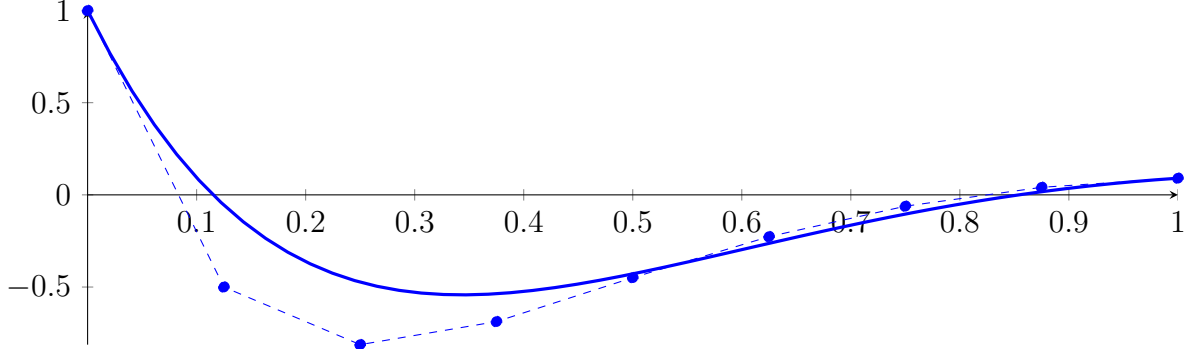
Longest intersection interval: 1

⇒ Bisection: first half [0, 0.25] und second half [0.25, 0.5]

### 2.3 Recursion Branch 1 1 1 on the First Half [0, 0.25]

Normalized monomial und Bézier representations and the Bézier polygon:

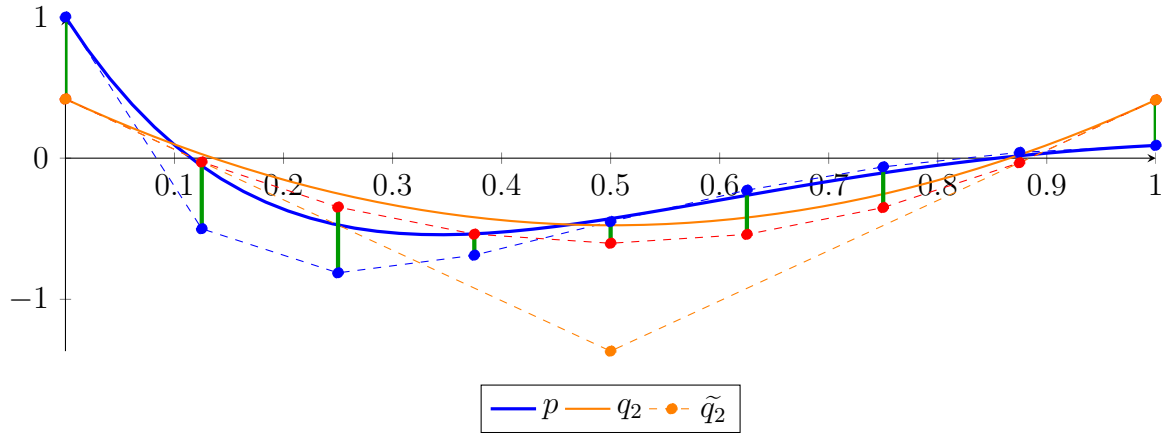
$$\begin{aligned}
 p &= 0.0323181X^8 - 0.508301X^7 + 3.41797X^6 - 12.9063X^5 + 29.8047X^4 - 42X^3 + 33.25X^2 - 12X + 1 \\
 &= 1B_{0,8}(X) - 0.5B_{1,8}(X) - 0.8125B_{2,8}(X) - 0.6875B_{3,8}(X) - 0.449219B_{4,8}(X) \\
 &\quad - 0.226563B_{5,8}(X) - 0.0615234B_{6,8}(X) + 0.0409546B_{7,8}(X) + 0.0904236B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= 3.56571X^2 - 3.57271X + 0.419352 \\
 &= 0.419352B_{0,2} - 1.36701B_{1,2} + 0.412345B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 2.04699 \cdot 10^{-13}X^8 - 8.68177 \cdot 10^{-13}X^7 + 1.49478 \cdot 10^{-12}X^6 - 1.32492 \cdot 10^{-12}X^5 \\
 &\quad + 6.26609 \cdot 10^{-13}X^4 - 1.42921 \cdot 10^{-13}X^3 + 3.56571X^2 - 3.57271X + 0.419352 \\
 &= 0.419352B_{0,8} - 0.0272376B_{1,8} - 0.34648B_{2,8} - 0.538376B_{3,8} - 0.602925B_{4,8} \\
 &\quad - 0.540128B_{5,8} - 0.349984B_{6,8} - 0.0324927B_{7,8} + 0.412345B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.580648$ .

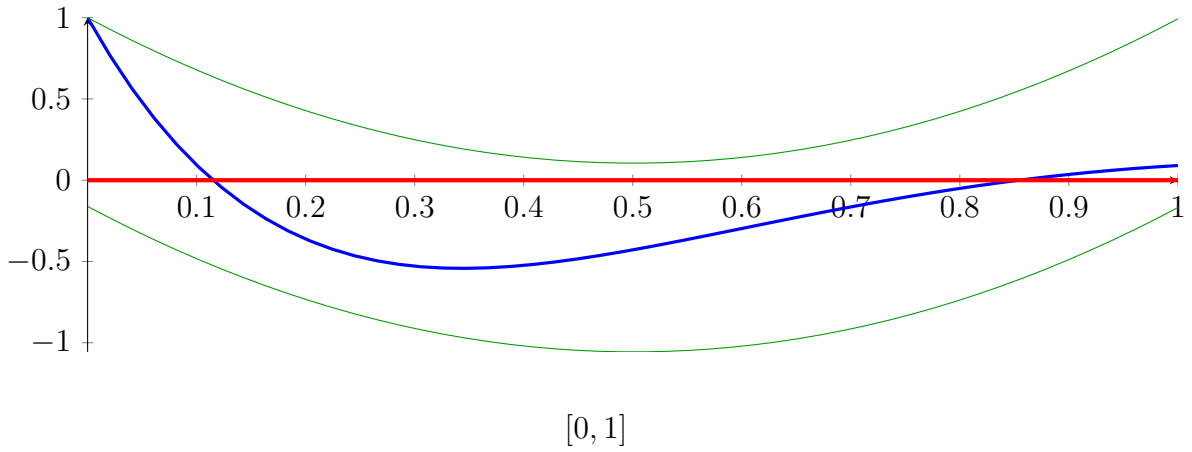
Bounding polynomials  $M$  and  $m$ :

$$\begin{aligned}
 M &= 3.56571X^2 - 3.57271X + 1 \\
 m &= 3.56571X^2 - 3.57271X - 0.161297
 \end{aligned}$$

Root of  $M$  and  $m$ :

$$N(M) = \{ \} \qquad N(m) = \{ -0.0432775, 1.04524 \}$$

Intersection intervals:



Longest intersection interval: 1

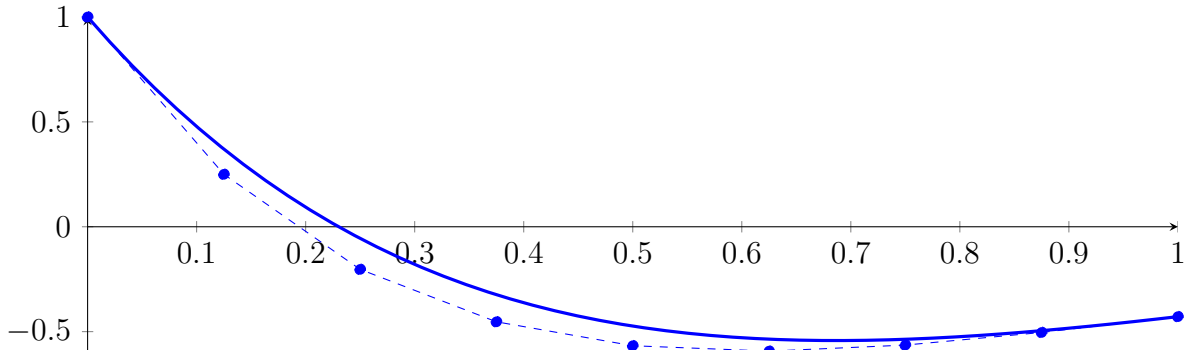
⇒ Bisection: first half  $[0, 0.125]$  und second half  $[0.125, 0.25]$

Bisection point is very near to a root?!?

## 2.4 Recursion Branch 1 1 1 1 on the First Half $[0, 0.125]$

Normalized monomial und Bézier representations and the Bézier polygon:

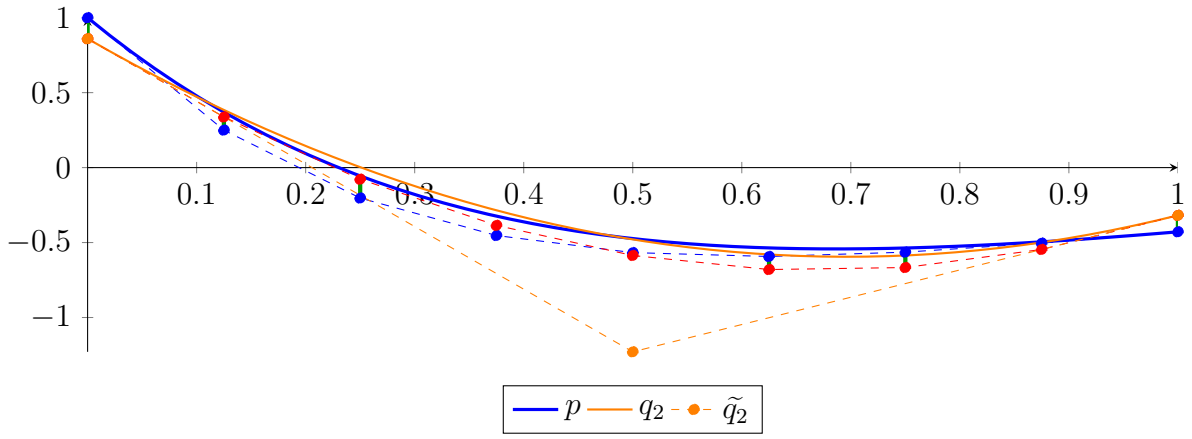
$$\begin{aligned}
 p &= 0.000126243X^8 - 0.0039711X^7 + 0.0534058X^6 \\
 &\quad - 0.40332X^5 + 1.86279X^4 - 5.25X^3 + 8.3125X^2 - 6X + 1 \\
 &= 1B_{0,8}(X) + 0.25B_{1,8}(X) - 0.203125B_{2,8}(X) - 0.453125B_{3,8}(X) - 0.567139B_{4,8}(X) \\
 &\quad - 0.592896B_{5,8}(X) - 0.564011B_{6,8}(X) - 0.503869B_{7,8}(X) - 0.428466B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= 2.99928X^2 - 4.17755X + 0.859833 \\
 &= 0.859833B_{0,2} - 1.22894B_{1,2} - 0.318436B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 4.76325 \cdot 10^{-13}X^8 - 1.88239 \cdot 10^{-12}X^7 + 3.02234 \cdot 10^{-12}X^6 - 2.51603 \cdot 10^{-12}X^5 \\
 &\quad + 1.13911 \cdot 10^{-12}X^4 - 2.61844 \cdot 10^{-13}X^3 + 2.99928X^2 - 4.17755X + 0.859833 \\
 &= 0.859833B_{0,8} + 0.337639B_{1,8} - 0.0774367B_{2,8} - 0.385396B_{3,8} - 0.586238B_{4,8} \\
 &\quad - 0.679963B_{5,8} - 0.666571B_{6,8} - 0.546062B_{7,8} - 0.318436B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.140167$ .

**Bounding polynomials  $M$  and  $m$ :**

$$M = 2.99928X^2 - 4.17755X + 1$$

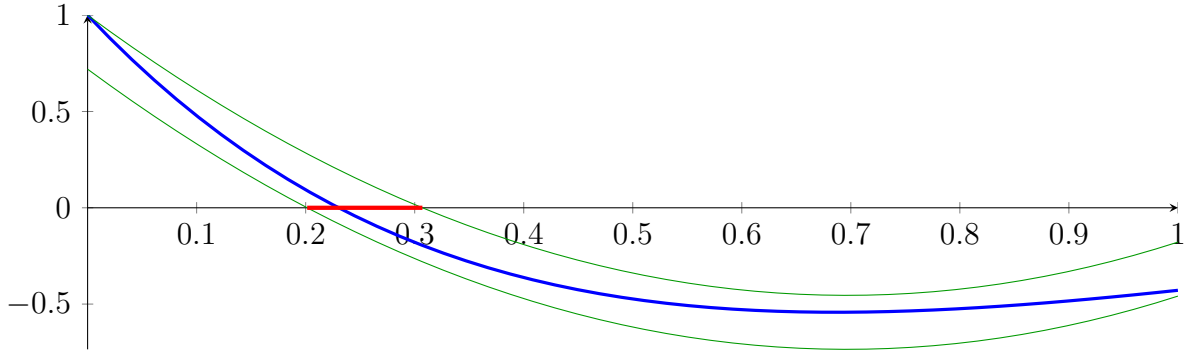
$$m = 2.99928X^2 - 4.17755X + 0.719665$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{0.307074, 1.08578\}$$

$$N(m) = \{0.201388, 1.19146\}$$

**Intersection intervals:**



$$[0.201388, 0.307074]$$

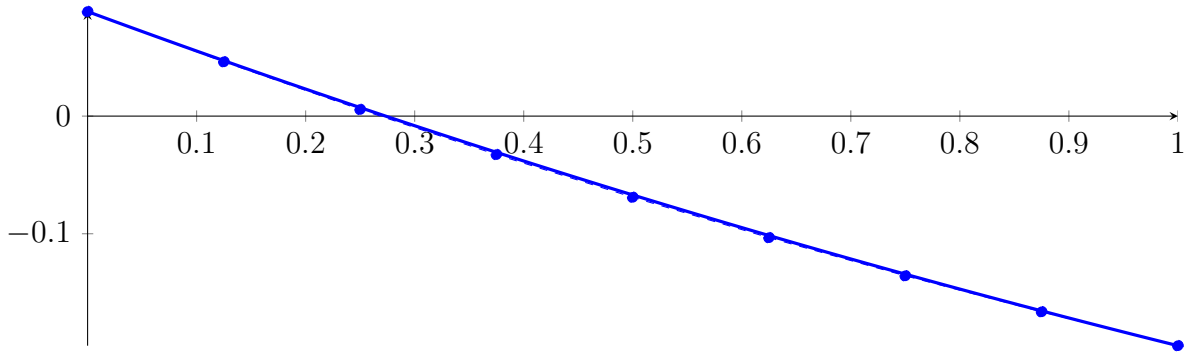
Longest intersection interval: 0.105686

$\implies$  Selective recursion: interval 1:  $[0.0251735, 0.0383842]$ ,

## 2.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.0251735, 0.0383842]$

**Normalized monomial und Bézier representations and the Bézier polygon:**

$$\begin{aligned}
p &= 1.96492 \cdot 10^{-12} X^8 - 5.54878 \cdot 10^{-10} X^7 + 6.68191 \cdot 10^{-08} X^6 - 4.51082 \cdot 10^{-06} X^5 \\
&\quad + 0.000185645 X^4 - 0.00460911 X^3 + 0.062128 X^2 - 0.341688 X + 0.0888568 \\
&= 0.0888568 B_{0,8}(X) + 0.0461458 B_{1,8}(X) + 0.00565363 B_{2,8}(X) - 0.032702 B_{3,8}(X) \\
&\quad - 0.0690007 B_{4,8}(X) - 0.10332 B_{5,8}(X) - 0.135733 B_{6,8}(X) - 0.166314 B_{7,8}(X) - 0.195131 B_{8,8}(X)
\end{aligned}$$



**Degree reduction and raising:**

$$q_2 = 0.0555247X^2 - 0.339088X + 0.0886418$$

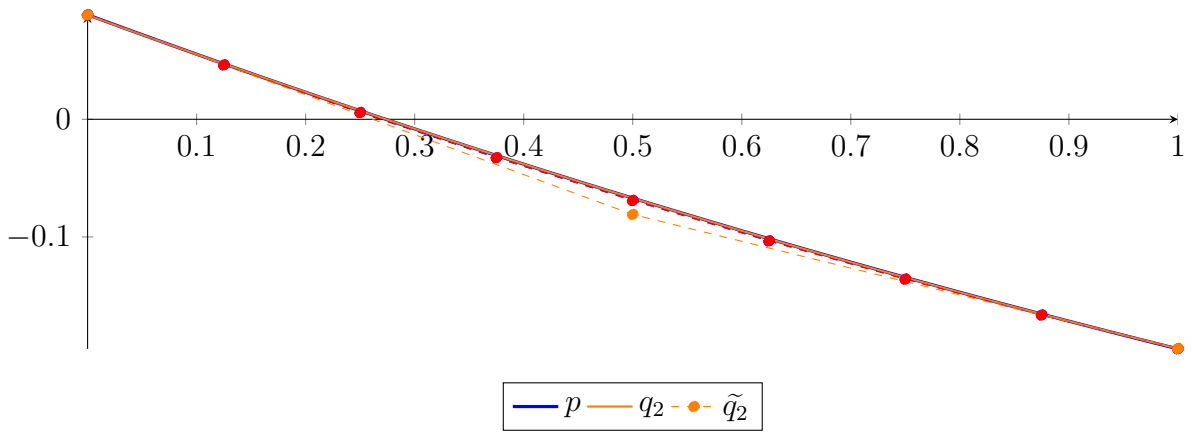
$$= 0.0886418B_{0,2} - 0.0809021B_{1,2} - 0.194921B_{2,2}$$

$$\tilde{q}_2 = 1.70903 \cdot 10^{-13}X^8 - 6.70628 \cdot 10^{-13}X^7 + 1.07189 \cdot 10^{-12}X^6 - 8.95006 \cdot 10^{-13}X^5$$

$$+ 4.1449 \cdot 10^{-13}X^4 - 1.03011 \cdot 10^{-13}X^3 + 0.0555247X^2 - 0.339088X + 0.0886418$$

$$= 0.0886418B_{0,8} + 0.0462559B_{1,8} + 0.0058529B_{2,8} - 0.032567B_{3,8} - 0.0690039B_{4,8}$$

$$- 0.103458B_{5,8} - 0.135929B_{6,8} - 0.166416B_{7,8} - 0.194921B_{8,8}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.000215018$ .

**Bounding polynomials  $M$  and  $m$ :**

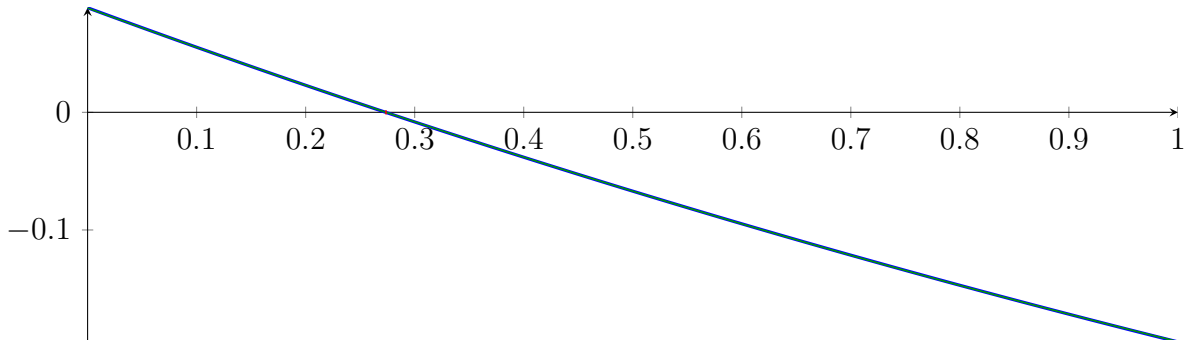
$$M = 0.0555247X^2 - 0.339088X + 0.0888568$$

$$m = 0.0555247X^2 - 0.339088X + 0.0884268$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{0.274374, 5.8326\} \quad N(m) = \{0.272981, 5.83399\}$$

**Intersection intervals:**



$$[0.272981, 0.274374]$$

Longest intersection interval: 0.00139307

$\implies$  Selective recursion: interval 1:  $[0.0287798, 0.0287982]$ ,

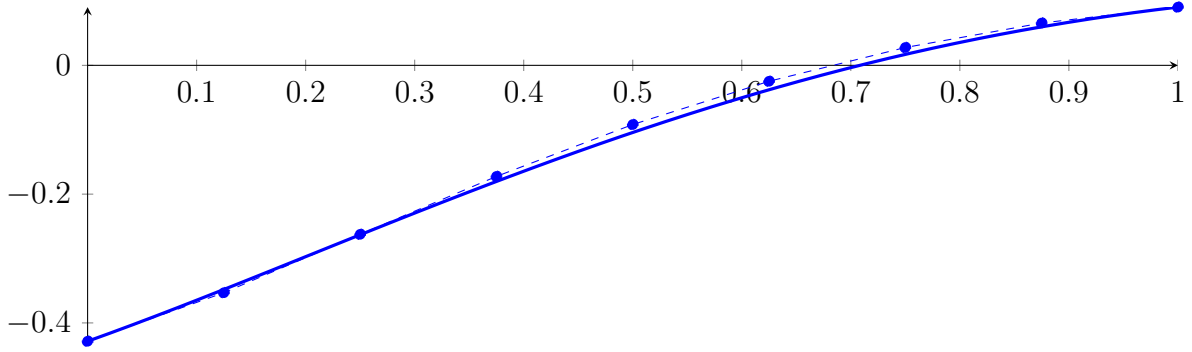
## 2.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.0287798, 0.0287982]

Found root in interval [0.0287798, 0.0287982] at recursion depth 6!

## 2.7 Recursion Branch 1 1 1 2 on the Second Half [0.125, 0.25]

Normalized monomial und Bézier representations and the Bézier polygon:

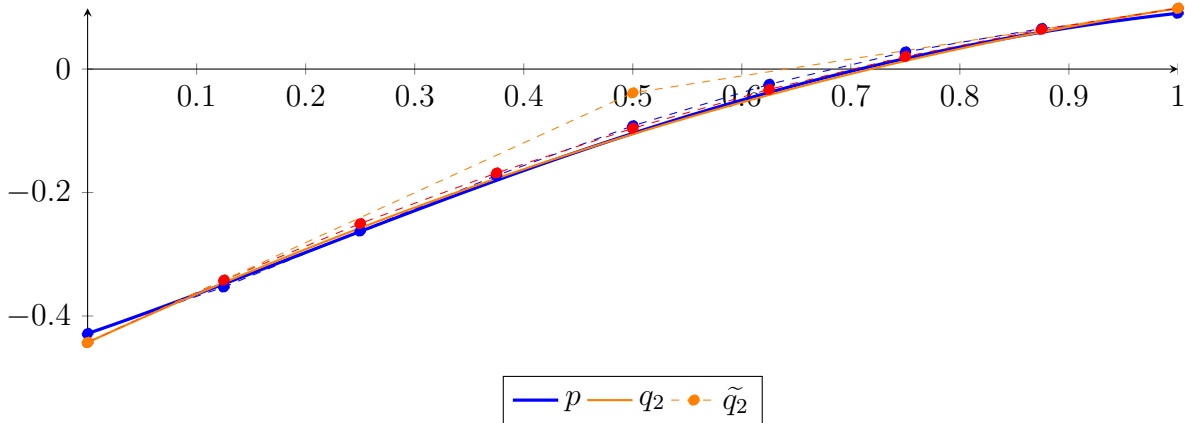
$$\begin{aligned}
 p &= 0.000126243X^8 - 0.00296116X^7 + 0.0291429X^6 - 0.159209X^5 \\
 &\quad + 0.517126X^4 - 0.895835X^3 + 0.427283X^2 + 0.603217X - 0.428466 \\
 &= -0.428466B_{0,8}(X) - 0.353064B_{1,8}(X) - 0.262402B_{2,8}(X) - 0.172477B_{3,8}(X) - 0.091898B_{4,8}(X) \\
 &\quad - 0.0247307B_{5,8}(X) + 0.0277023B_{6,8}(X) + 0.0656891B_{7,8}(X) + 0.0904236B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -0.267197X^2 + 0.8085X - 0.442876 \\
 &= -0.442876B_{0,2} - 0.038626B_{1,2} + 0.0984275B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 2.57314 \cdot 10^{-13}X^8 - 1.06329 \cdot 10^{-12}X^7 + 1.8005 \cdot 10^{-12}X^6 - 1.60212 \cdot 10^{-12}X^5 \\
 &\quad + 7.96456 \cdot 10^{-13}X^4 - 2.15712 \cdot 10^{-13}X^3 - 0.267197X^2 + 0.8085X - 0.442876 \\
 &= -0.442876B_{0,8} - 0.341813B_{1,8} - 0.250294B_{2,8} - 0.168317B_{3,8} - 0.0958824B_{4,8} \\
 &\quad - 0.0329908B_{5,8} + 0.020358B_{6,8} + 0.0641641B_{7,8} + 0.0984275B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.0144095$ .

Bounding polynomials  $M$  and  $m$ :

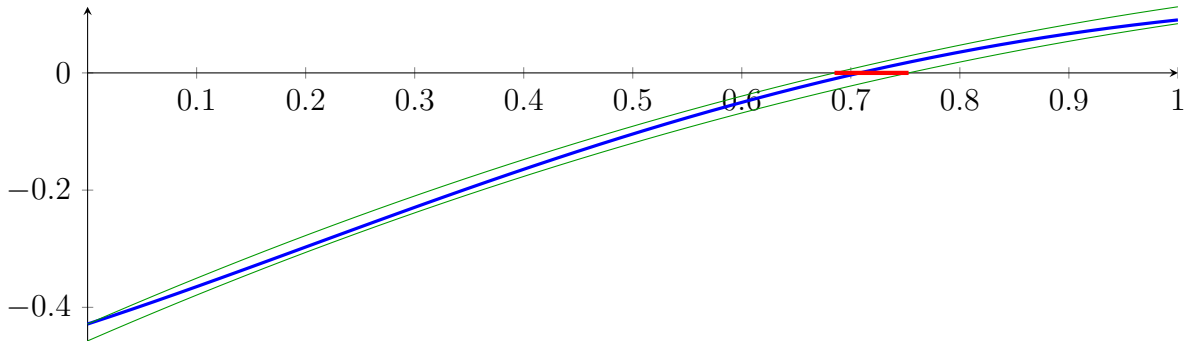
$$\begin{aligned}
 M &= -0.267197X^2 + 0.8085X - 0.428466 \\
 m &= -0.267197X^2 + 0.8085X - 0.457285
 \end{aligned}$$

Root of  $M$  and  $m$ :

$$N(M) = \{0.685044, 2.34082\} \qquad N(m) = \{0.75297, 2.27289\}$$

Intersection intervals:





[0.685044, 0.75297]

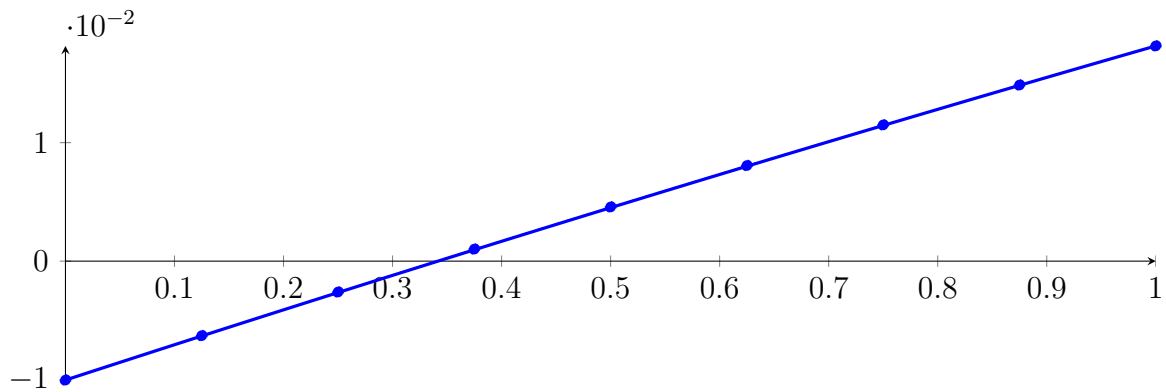
Longest intersection interval: 0.0679263

⇒ Selective recursion: interval 1: [0.21063, 0.219121],

## 2.8 Recursion Branch 1 1 1 2 1 in Interval 1: [0.21063, 0.219121]

Normalized monomial und Bézier representations and the Bézier polygon:

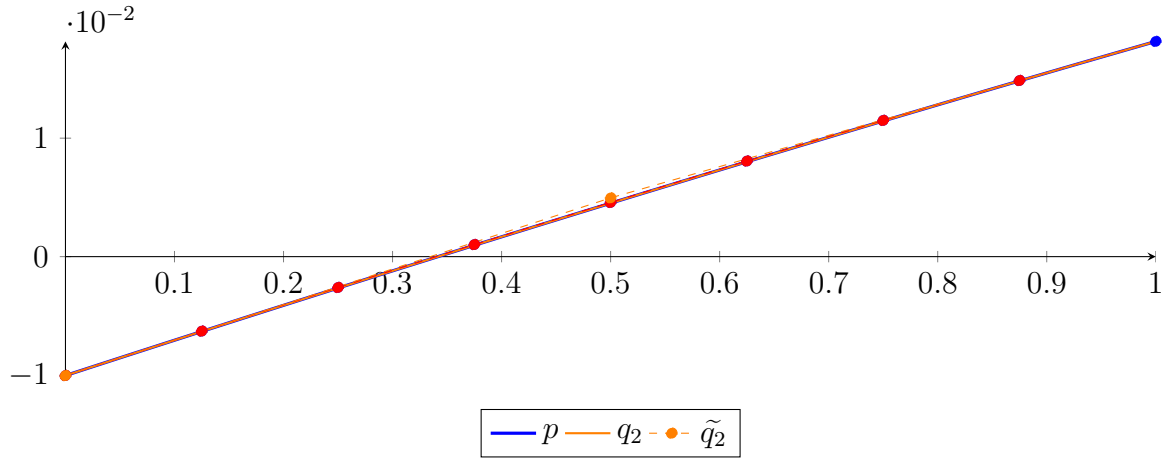
$$\begin{aligned}
 p &= 5.72143 \cdot 10^{-14} X^8 - 1.51411 \cdot 10^{-11} X^7 + 1.63075 \cdot 10^{-09} X^6 - 9.59233 \cdot 10^{-08} X^5 \\
 &\quad + 3.09905 \cdot 10^{-06} X^4 - 1.89125 \cdot 10^{-05} X^3 - 0.00176378 X^2 + 0.0299832 X - 0.0100376 \\
 &= -0.0100376 B_{0,8}(X) - 0.00628973 B_{1,8}(X) - 0.00260482 B_{2,8}(X) \\
 &\quad + 0.00101676 B_{3,8}(X) + 0.00457472 B_{4,8}(X) + 0.0080688 B_{5,8}(X) \\
 &\quad + 0.0114988 B_{6,8}(X) + 0.0148645 B_{7,8}(X) + 0.0181659 B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -0.00178701 X^2 + 0.0299918 X - 0.0100383 \\
 &= -0.0100383 B_{0,2} + 0.0049576 B_{1,2} + 0.0181665 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -1.29309 \cdot 10^{-14} X^8 + 5.03053 \cdot 10^{-14} X^7 - 7.96032 \cdot 10^{-14} X^6 + 6.57208 \cdot 10^{-14} X^5 \\
 &\quad - 3.00638 \cdot 10^{-14} X^4 + 7.37307 \cdot 10^{-15} X^3 - 0.00178701 X^2 + 0.0299918 X - 0.0100383 \\
 &= -0.0100383 B_{0,8} - 0.00628934 B_{1,8} - 0.00260418 B_{2,8} + 0.00101715 B_{3,8} \\
 &\quad + 0.00457467 B_{4,8} + 0.00806836 B_{5,8} + 0.0114982 B_{6,8} + 0.0148643 B_{7,8} + 0.0181665 B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 6.90075 \cdot 10^{-07}$ .

**Bounding polynomials  $M$  and  $m$ :**

$$M = -0.00178701X^2 + 0.0299918X - 0.0100376$$

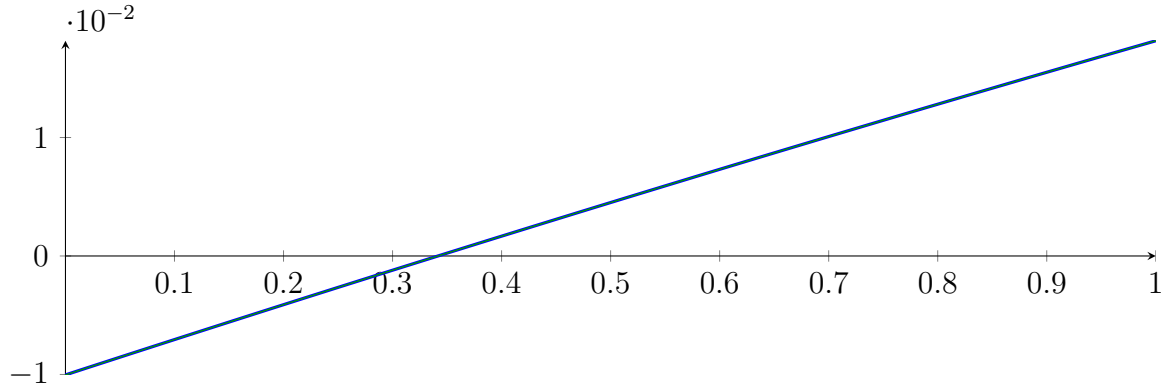
$$m = -0.00178701X^2 + 0.0299918X - 0.010039$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{0.341633, 16.4416\}$$

$$N(m) = \{0.341681, 16.4416\}$$

**Intersection intervals:**



$$[0.341633, 0.341681]$$

Longest intersection interval:  $4.79706 \cdot 10^{-05}$

$\implies$  Selective recursion: [interval 1: \[0.213531, 0.213532\]](#),

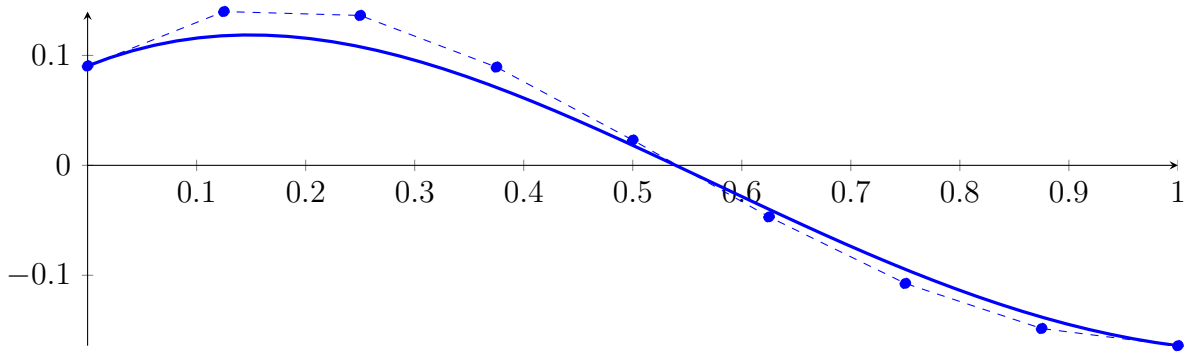
## 2.9 Recursion Branch 1 1 1 2 1 1 in Interval 1: [0.213531, 0.213532]

Found root in interval [0.213531, 0.213532] at recursion depth 6!

## 2.10 Recursion Branch 1 1 2 on the Second Half [0.25, 0.5]

**Normalized monomial und Bézier representations and the Bézier polygon:**

$$\begin{aligned} p &= 0.0323181X^8 - 0.249756X^7 + 0.764771X^6 - 1.26294X^5 \\ &\quad + 1.01471X^4 + 0.534912X^3 - 1.48425X^2 + 0.395752X + 0.0904236 \\ &= 0.0904236B_{0,8}(X) + 0.139893B_{1,8}(X) + 0.136353B_{2,8}(X) + 0.0893555B_{3,8}(X) + 0.0229492B_{4,8}(X) \\ &\quad - 0.046875B_{5,8}(X) - 0.107422B_{6,8}(X) - 0.148438B_{7,8}(X) - 0.164063B_{8,8}(X) \end{aligned}$$



**Degree reduction and raising:**

$$q_2 = -0.2142X^2 - 0.120028X + 0.132767$$

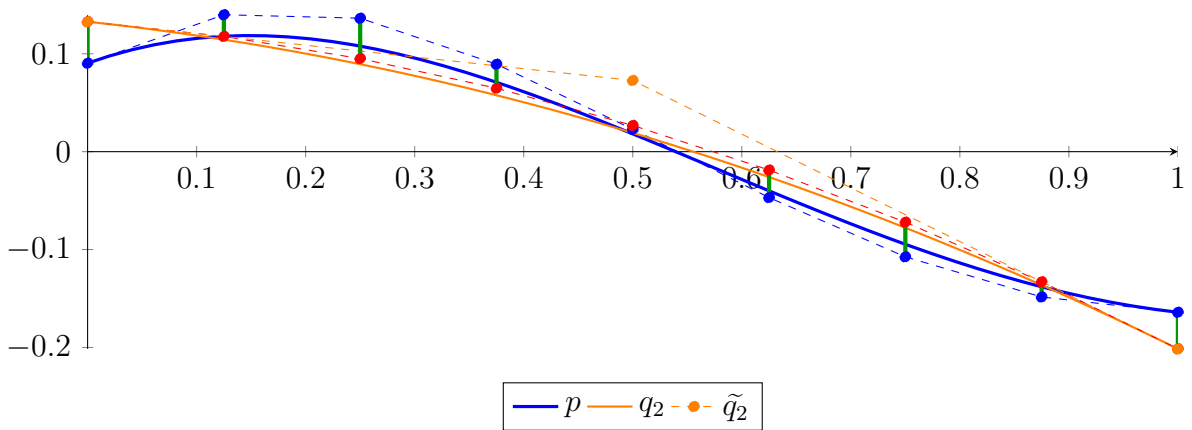
$$= 0.132767B_{0,2} + 0.072753B_{1,2} - 0.201461B_{2,2}$$

$$\tilde{q}_2 = 4.37953 \cdot 10^{-14}X^8 - 1.53225 \cdot 10^{-13}X^7 + 2.11155 \cdot 10^{-13}X^6 - 1.45809 \cdot 10^{-13}X^5$$

$$+ 5.33118 \cdot 10^{-14}X^4 - 1.01379 \cdot 10^{-14}X^3 - 0.2142X^2 - 0.120028X + 0.132767$$

$$= 0.132767B_{0,8} + 0.117764B_{1,8} + 0.0951101B_{2,8} + 0.0648065B_{3,8} + 0.026853B_{4,8}$$

$$- 0.0187506B_{5,8} - 0.0720042B_{6,8} - 0.132908B_{7,8} - 0.201461B_{8,8}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.0423436$ .

**Bounding polynomials  $M$  and  $m$ :**

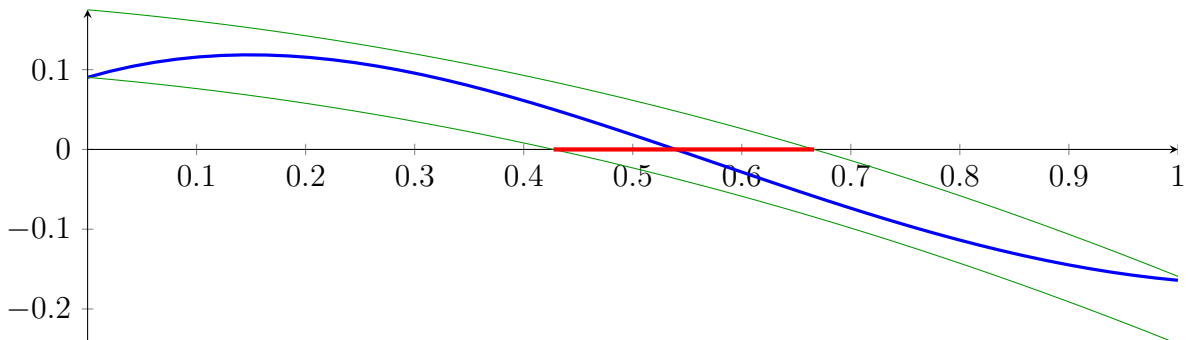
$$M = -0.2142X^2 - 0.120028X + 0.175111$$

$$m = -0.2142X^2 - 0.120028X + 0.0904236$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{-1.22676, 0.6664\} \quad N(m) = \{-0.987741, 0.427385\}$$

**Intersection intervals:**



$$[0.427385, 0.6664]$$

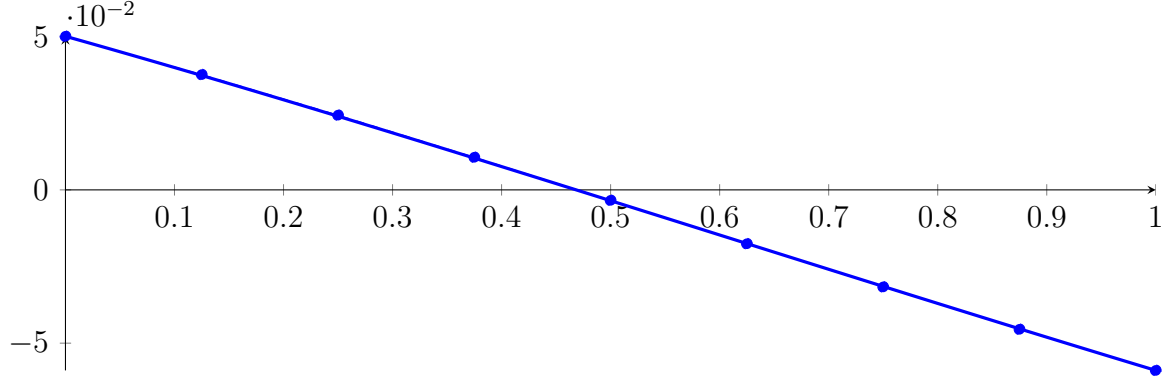
Longest intersection interval: 0.239015

$\implies$  Selective recursion: interval 1:  $[0.356846, 0.4166]$ ,

## 2.11 Recursion Branch 1 1 2 1 in Interval 1: [0.356846, 0.4166]

Normalized monomial und Bézier representations and the Bézier polygon:

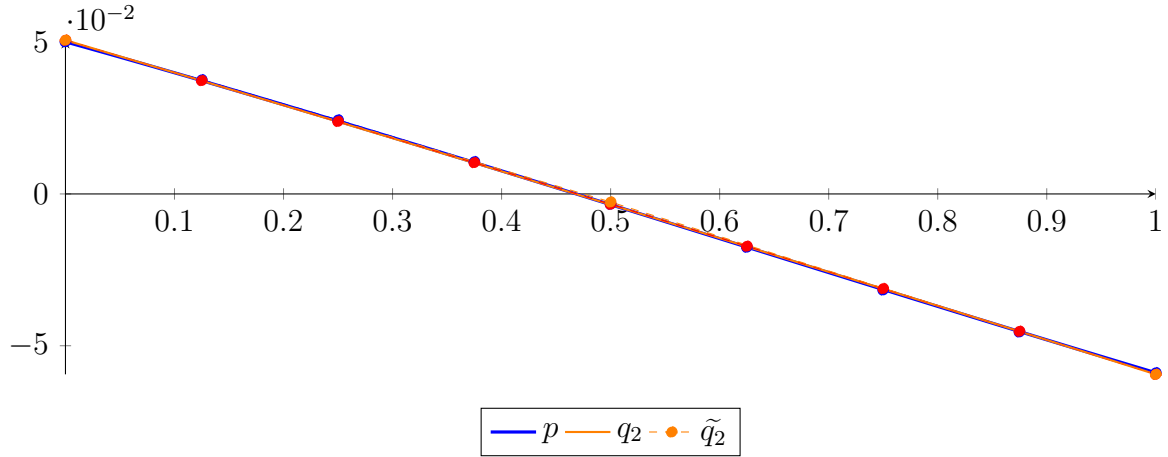
$$\begin{aligned}
 p &= 3.44232 \cdot 10^{-07} X^8 - 6.20581 \cdot 10^{-06} X^7 + 3.40947 \cdot 10^{-05} X^6 - 9.2488 \cdot 10^{-05} X^5 \\
 &\quad - 0.000638538 X^4 + 0.0121653 X^3 - 0.0204976 X^2 - 0.0999572 X + 0.0501019 \\
 &= 0.0501019 B_{0,8}(X) + 0.0376073 B_{1,8}(X) + 0.0243805 B_{2,8}(X) \\
 &\quad + 0.010639 B_{3,8}(X) - 0.00340921 B_{4,8}(X) - 0.0175668 B_{5,8}(X) \\
 &\quad - 0.0316476 B_{6,8}(X) - 0.0454767 B_{7,8}(X) - 0.0588904 B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -0.00345878 X^2 - 0.106606 X + 0.0506489 \\
 &= 0.0506489 B_{0,2} - 0.00265396 B_{1,2} - 0.0594156 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 2.03423 \cdot 10^{-14} X^8 - 7.51159 \cdot 10^{-14} X^7 + 1.11345 \cdot 10^{-13} X^6 - 8.47406 \cdot 10^{-14} X^5 \\
 &\quad + 3.50083 \cdot 10^{-14} X^4 - 7.49663 \cdot 10^{-15} X^3 - 0.00345878 X^2 - 0.106606 X + 0.0506489 \\
 &= 0.0506489 B_{0,8} + 0.0373232 B_{1,8} + 0.0238739 B_{2,8} + 0.0103012 B_{3,8} - 0.00339513 B_{4,8} \\
 &\quad - 0.0172149 B_{5,8} - 0.0311583 B_{6,8} - 0.0452252 B_{7,8} - 0.0594156 B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.000546952$ .

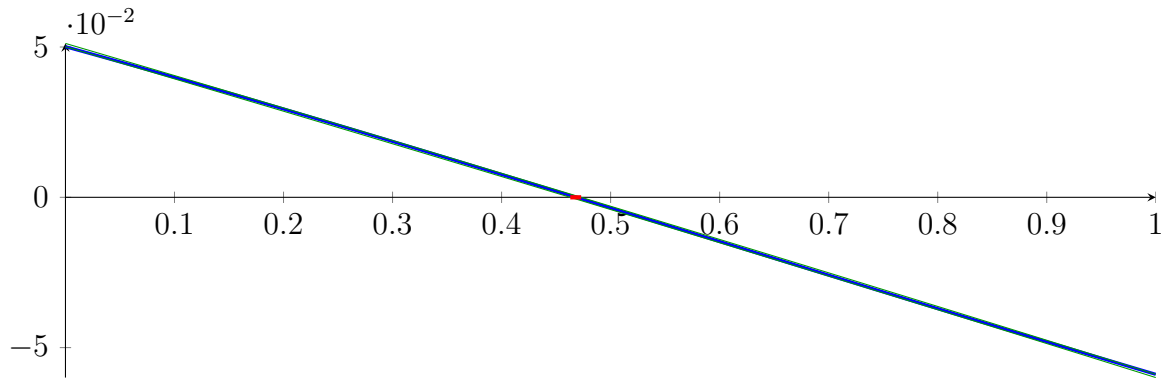
Bounding polynomials  $M$  and  $m$ :

$$\begin{aligned}
 M &= -0.00345878 X^2 - 0.106606 X + 0.0511958 \\
 m &= -0.00345878 X^2 - 0.106606 X + 0.0501019
 \end{aligned}$$

Root of  $M$  and  $m$ :

$$N(M) = \{-31.2947, 0.472977\} \qquad N(m) = \{-31.2848, 0.463019\}$$

Intersection intervals:



[0.463019, 0.472977]

Longest intersection interval: 0.00995879

⇒ Selective recursion: interval 1: [0.384513, 0.385108],

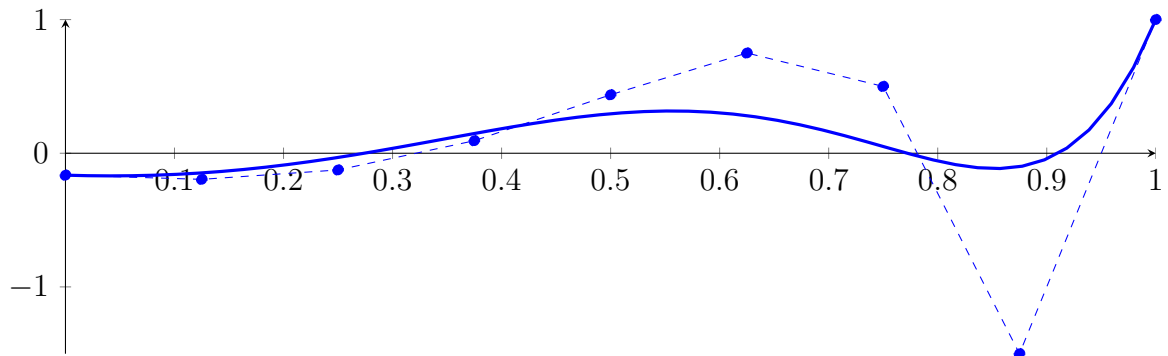
## 2.12 Recursion Branch 1 1 2 1 1 in Interval 1: [0.384513, 0.385108]

Found root in interval [0.384513, 0.385108] at recursion depth 5!

## 2.13 Recursion Branch 1 2 on the Second Half [0.5, 1]

Normalized monomial und Bézier representations and the Bézier polygon:

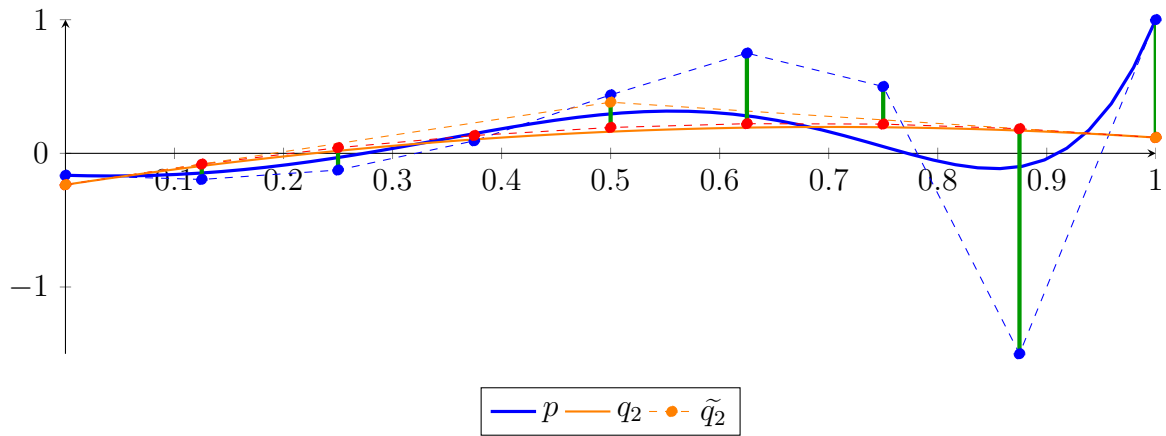
$$\begin{aligned}
 p &= 8.27344X^8 + 1.125X^7 - 5.03125X^6 - 3.5X^5 - 4.92188X^4 + 2.625X^3 + 2.84375X^2 - 0.25X - 0.164063 \\
 &= -0.164063B_{0,8}(X) - 0.195313B_{1,8}(X) - 0.125B_{2,8}(X) + 0.09375B_{3,8}(X) \\
 &\quad + 0.4375B_{4,8}(X) + 0.75B_{5,8}(X) + 0.5B_{6,8}(X) - 1.5B_{7,8}(X) + 1B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -0.882102X^2 + 1.23523X - 0.235038 \\
 &= -0.235038B_{0,2} + 0.382576B_{1,2} + 0.118087B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -1.89929 \cdot 10^{-13}X^8 + 7.53552 \cdot 10^{-13}X^7 - 1.21661 \cdot 10^{-12}X^6 + 1.02152 \cdot 10^{-12}X^5 \\
 &\quad - 4.69668 \cdot 10^{-13}X^4 + 1.11813 \cdot 10^{-13}X^3 - 0.882102X^2 + 1.23523X - 0.235038 \\
 &= -0.235038B_{0,8} - 0.0806345B_{1,8} + 0.0422653B_{2,8} + 0.133661B_{3,8} \\
 &\quad + 0.193554B_{4,8} + 0.221943B_{5,8} + 0.218828B_{6,8} + 0.184209B_{7,8} + 0.118087B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 1.68421$ .

**Bounding polynomials  $M$  and  $m$ :**

$$M = -0.882102X^2 + 1.23523X + 1.44917$$

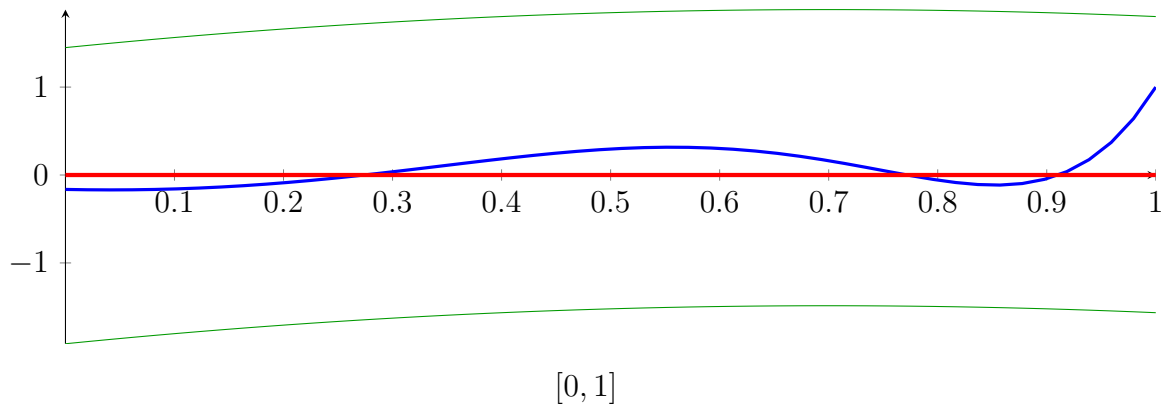
$$m = -0.882102X^2 + 1.23523X - 1.91925$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{-0.760348, 2.16067\}$$

$$N(m) = \{\}$$

**Intersection intervals:**



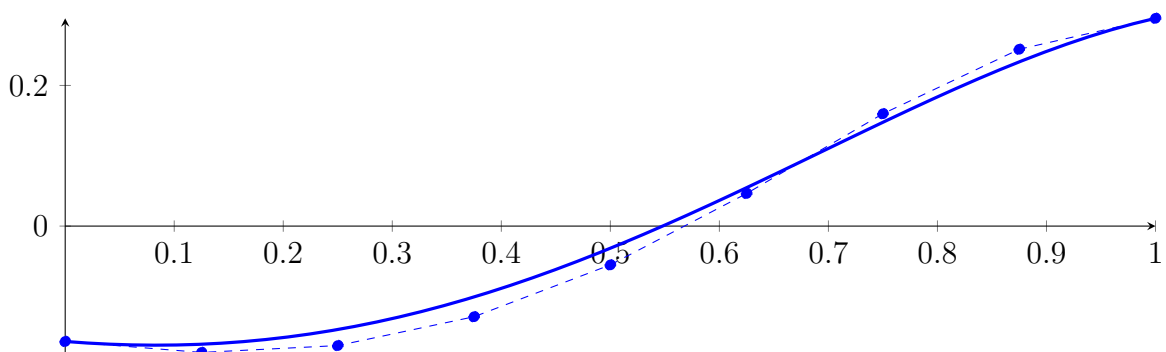
Longest intersection interval: 1

$\implies$  Bisection: first half  $[0.5, 0.75]$  und second half  $[0.75, 1]$

## 2.14 Recursion Branch 1 2 1 on the First Half $[0.5, 0.75]$

**Normalized monomial und Bézier representations and the Bézier polygon:**

$$\begin{aligned}
 p &= 0.0323181X^8 + 0.00878906X^7 - 0.0786133X^6 - 0.109375X^5 \\
 &\quad - 0.307617X^4 + 0.328125X^3 + 0.710937X^2 - 0.125X - 0.164063 \\
 &= -0.164063B_{0,8}(X) - 0.179688B_{1,8}(X) - 0.169922B_{2,8}(X) - 0.128906B_{3,8}(X) - 0.0551758B_{4,8}(X) \\
 &\quad + 0.0463867B_{5,8}(X) + 0.15979B_{6,8}(X) + 0.251465B_{7,8}(X) + 0.295502B_{8,8}(X)
 \end{aligned}$$



### Degree reduction and raising:

$$q_2 = 0.410312X^2 + 0.118546X - 0.189889$$

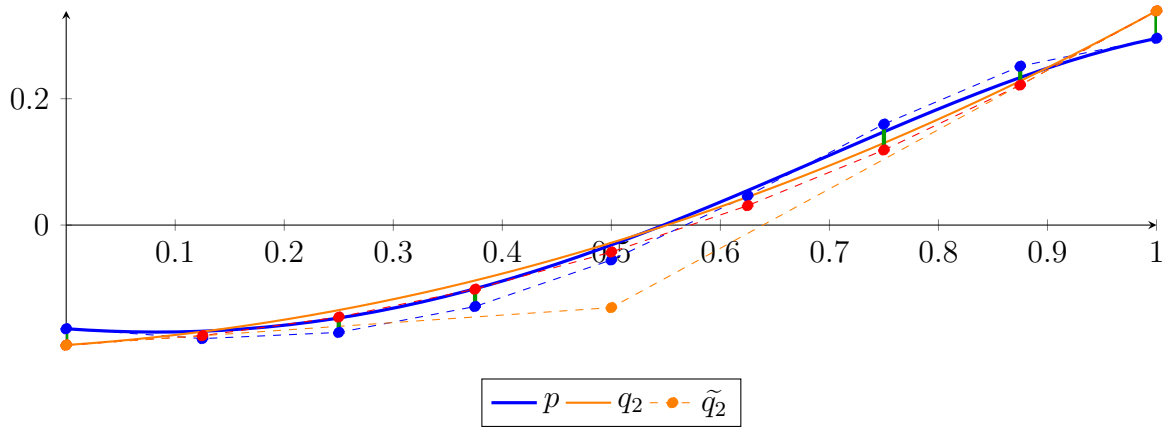
$$= -0.189889B_{0,2} - 0.130616B_{1,2} + 0.338968B_{2,2}$$

$$\tilde{q}_2 = -9.28149 \cdot 10^{-14}X^8 + 3.36056 \cdot 10^{-13}X^7 - 4.86486 \cdot 10^{-13}X^6 + 3.61157 \cdot 10^{-13}X^5$$

$$- 1.47057 \cdot 10^{-13}X^4 + 3.26762 \cdot 10^{-14}X^3 + 0.410312X^2 + 0.118546X - 0.189889$$

$$= -0.189889B_{0,8} - 0.175071B_{1,8} - 0.145599B_{2,8} - 0.101472B_{3,8} - 0.0426922B_{4,8}$$

$$+ 0.0307419B_{5,8} + 0.11883B_{6,8} + 0.221572B_{7,8} + 0.338968B_{8,8}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.0434666$ .

### Bounding polynomials $M$ and $m$ :

$$M = 0.410312X^2 + 0.118546X - 0.146422$$

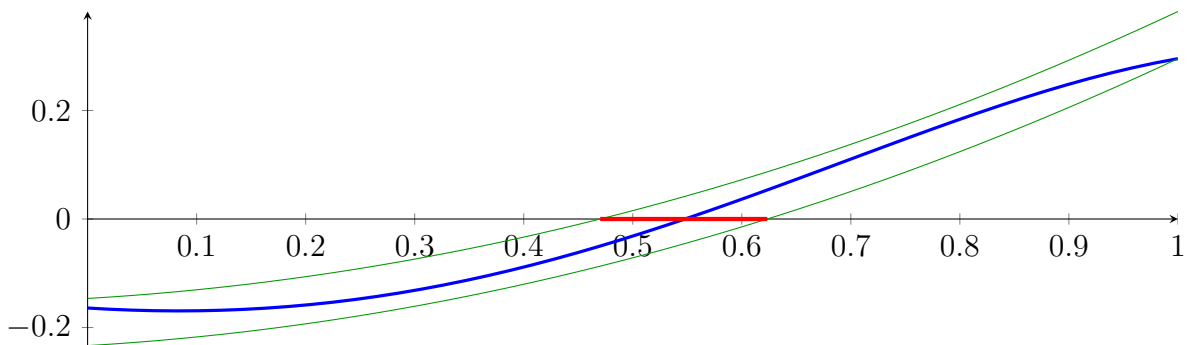
$$m = 0.410312X^2 + 0.118546X - 0.233356$$

### Root of $M$ and $m$ :

$$N(M) = \{-0.759051, 0.470135\}$$

$$N(m) = \{-0.91231, 0.623393\}$$

### Intersection intervals:



$$[0.470135, 0.623393]$$

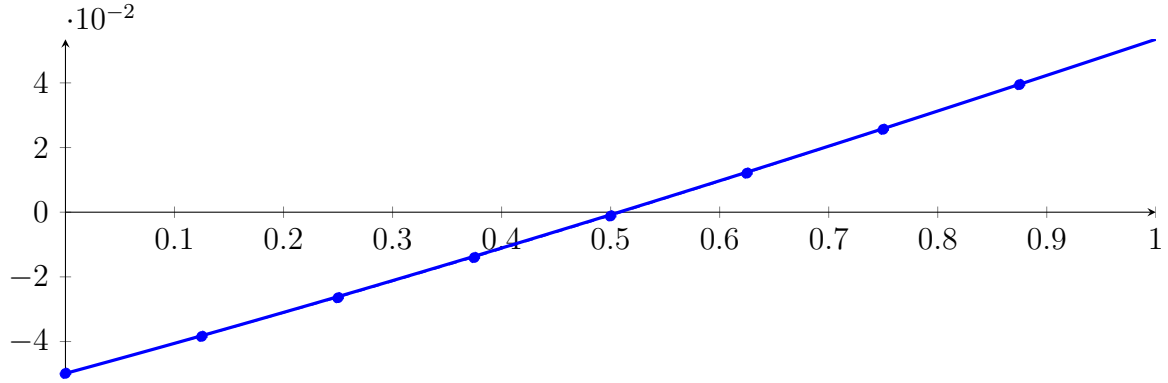
Longest intersection interval: 0.153258

$\implies$  Selective recursion: interval 1:  $[0.617534, 0.655848]$ ,

## 2.15 Recursion Branch 1 2 1 1 in Interval 1: [0.617534, 0.655848]

Normalized monomial und Bézier representations and the Bézier polygon:

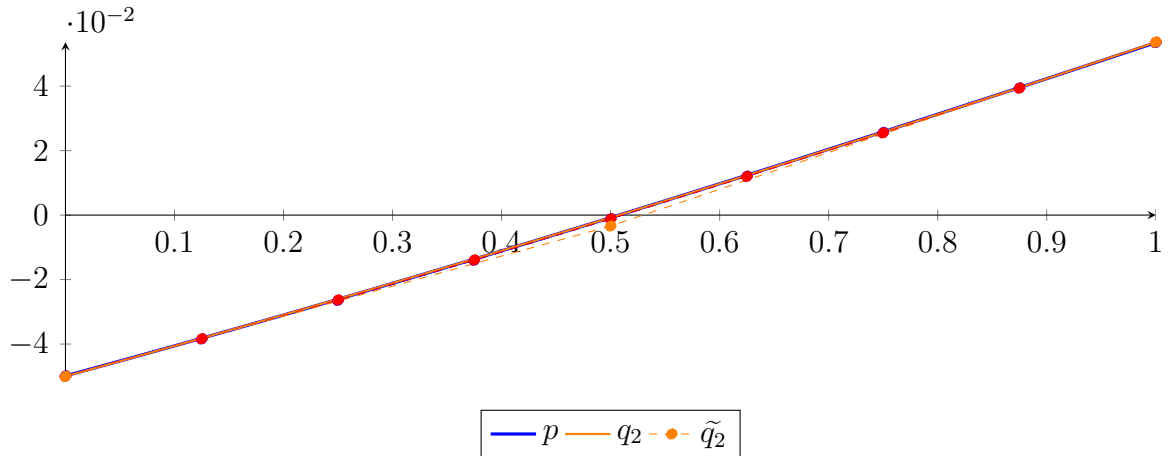
$$\begin{aligned}
 p &= 9.83654 \cdot 10^{-09} X^8 + 2.58851 \cdot 10^{-07} X^7 + 1.94789 \cdot 10^{-06} X^6 - 8.64718 \cdot 10^{-06} X^5 \\
 &\quad - 0.000376737 X^4 - 0.00215587 X^3 + 0.0142931 X^2 + 0.0915892 X - 0.049864 \\
 &= -0.049864 B_{0,8}(X) - 0.0384153 B_{1,8}(X) - 0.0264562 B_{2,8}(X) \\
 &\quad - 0.0140251 B_{3,8}(X) - 0.00116592 B_{4,8}(X) + 0.0120719 B_{5,8}(X) \\
 &\quad + 0.0256334 B_{6,8}(X) + 0.0394579 B_{7,8}(X) + 0.0534793 B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= 0.010402 X^2 + 0.0932339 X - 0.0500047 \\
 &= -0.0500047 B_{0,2} - 0.00338777 B_{1,2} + 0.0536312 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -1.1425 \cdot 10^{-14} X^8 + 3.95703 \cdot 10^{-14} X^7 - 5.35167 \cdot 10^{-14} X^6 + 3.55175 \cdot 10^{-14} X^5 \\
 &\quad - 1.17529 \cdot 10^{-14} X^4 + 1.60042 \cdot 10^{-15} X^3 + 0.010402 X^2 + 0.0932339 X - 0.0500047 \\
 &= -0.0500047 B_{0,8} - 0.0383505 B_{1,8} - 0.0263247 B_{2,8} - 0.0139275 B_{3,8} - 0.00115877 B_{4,8} \\
 &\quad + 0.0119815 B_{5,8} + 0.0254932 B_{6,8} + 0.0393764 B_{7,8} + 0.0536312 B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.000151851$ .

Bounding polynomials  $M$  and  $m$ :

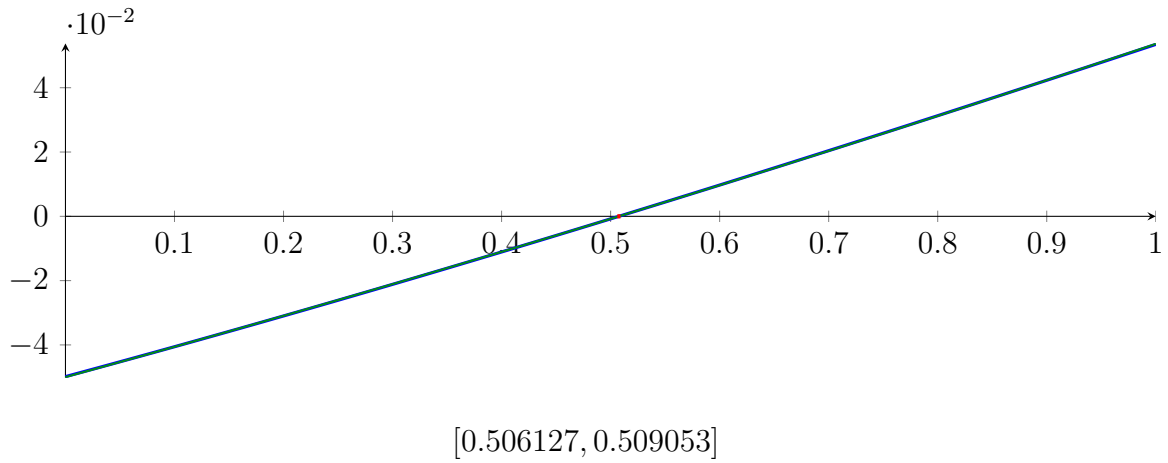
$$\begin{aligned}
 M &= 0.010402 X^2 + 0.0932339 X - 0.0498529 \\
 m &= 0.010402 X^2 + 0.0932339 X - 0.0501566
 \end{aligned}$$

Root of  $M$  and  $m$ :

$$N(M) = \{-9.46919, 0.506127\} \qquad N(m) = \{-9.47212, 0.509053\}$$

Intersection intervals:





Longest intersection interval: 0.00292601

⇒ Selective recursion: interval 1: [0.636926, 0.637038],

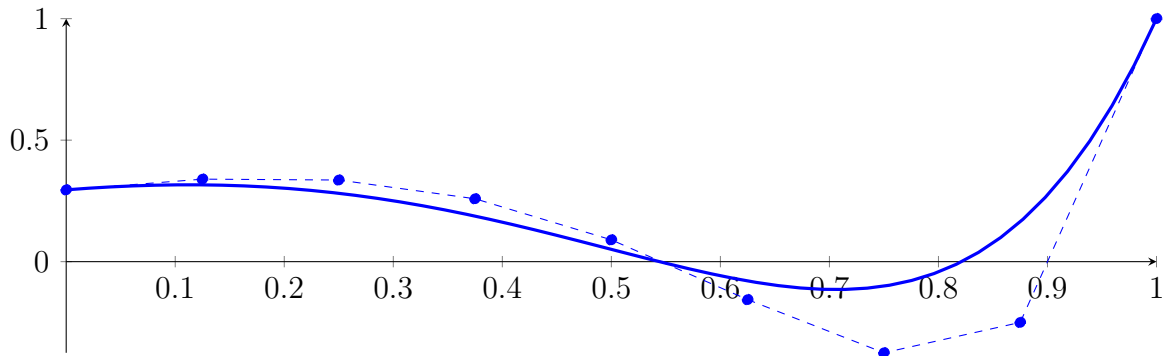
## 2.16 Recursion Branch 1 2 1 1 1 in Interval 1: [0.636926, 0.637038]

Found root in interval [0.636926, 0.637038] at recursion depth 5!

## 2.17 Recursion Branch 1 2 2 on the Second Half [0.75, 1]

Normalized monomial und Bézier representations and the Bézier polygon:

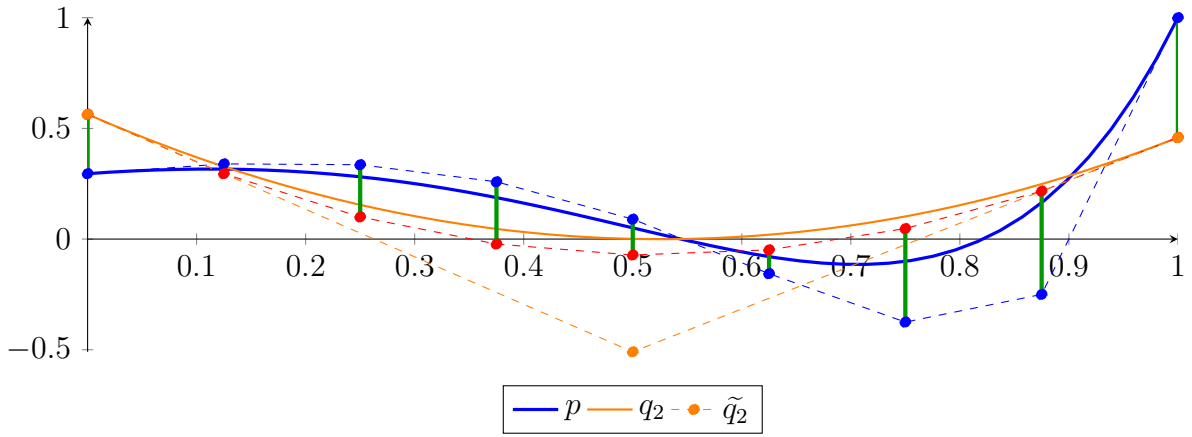
$$\begin{aligned}
 p &= 0.0323181X^8 + 0.267334X^7 + 0.887817X^6 + 1.41333X^5 \\
 &\quad + 0.536194X^4 - 1.45093X^3 - 1.33386X^2 + 0.352295X + 0.295502 \\
 &= 0.295502B_{0,8}(X) + 0.339539B_{1,8}(X) + 0.335937B_{2,8}(X) + 0.258789B_{3,8}(X) \\
 &\quad + 0.0898437B_{4,8}(X) - 0.15625B_{5,8}(X) - 0.375B_{6,8}(X) - 0.25B_{7,8}(X) + 1B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= 2.0408X^2 - 2.14581X + 0.563566 \\
 &= 0.563566B_{0,2} - 0.509339B_{1,2} + 0.458561B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -5.07578 \cdot 10^{-13}X^8 + 2.02023 \cdot 10^{-12}X^7 - 3.29055 \cdot 10^{-12}X^6 + 2.82209 \cdot 10^{-12}X^5 \\
 &\quad - 1.3632 \cdot 10^{-12}X^4 + 3.6591 \cdot 10^{-13}X^3 + 2.0408X^2 - 2.14581X + 0.563566 \\
 &= 0.563566B_{0,8} + 0.295339B_{1,8} + 0.0999999B_{2,8} - 0.0224554B_{3,8} - 0.072024B_{4,8} \\
 &\quad - 0.0487067B_{5,8} + 0.0474966B_{6,8} + 0.216586B_{7,8} + 0.458561B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.541439$ .

**Bounding polynomials  $M$  and  $m$ :**

$$M = 2.0408X^2 - 2.14581X + 1.105$$

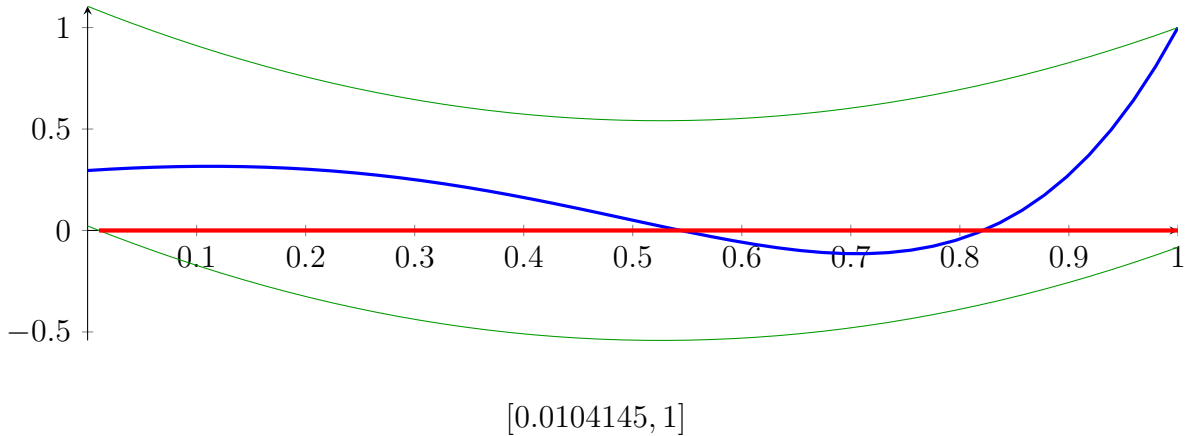
$$m = 2.0408X^2 - 2.14581X + 0.0221262$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{\}$$

$$N(m) = \{0.0104145, 1.04104\}$$

**Intersection intervals:**



$$[0.0104145, 1]$$

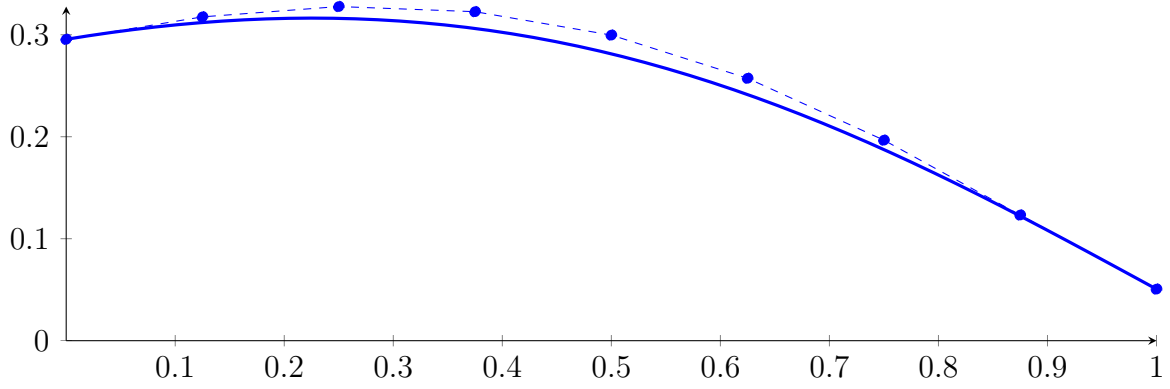
Longest intersection interval: 0.989586

$\implies$  Bisection: first half  $[0.75, 0.875]$  und second half  $[0.875, 1]$

## 2.18 Recursion Branch 1 2 2 1 on the First Half $[0.75, 0.875]$

**Normalized monomial und Bézier representations and the Bézier polygon:**

$$\begin{aligned}
 p &= 0.000126243X^8 + 0.00208855X^7 + 0.0138721X^6 + 0.0441666X^5 \\
 &\quad + 0.0335121X^4 - 0.181366X^3 - 0.333466X^2 + 0.176147X + 0.295502 \\
 &= 0.295502B_{0,8}(X) + 0.31752B_{1,8}(X) + 0.327629B_{2,8}(X) + 0.32259B_{3,8}(X) + 0.299643B_{4,8}(X) \\
 &\quad + 0.257295B_{5,8}(X) + 0.196605B_{6,8}(X) + 0.123225B_{7,8}(X) + 0.0505832B_{8,8}(X)
 \end{aligned}$$



**Degree reduction and raising:**

$$q_2 = -0.440555X^2 + 0.188569X + 0.295967$$

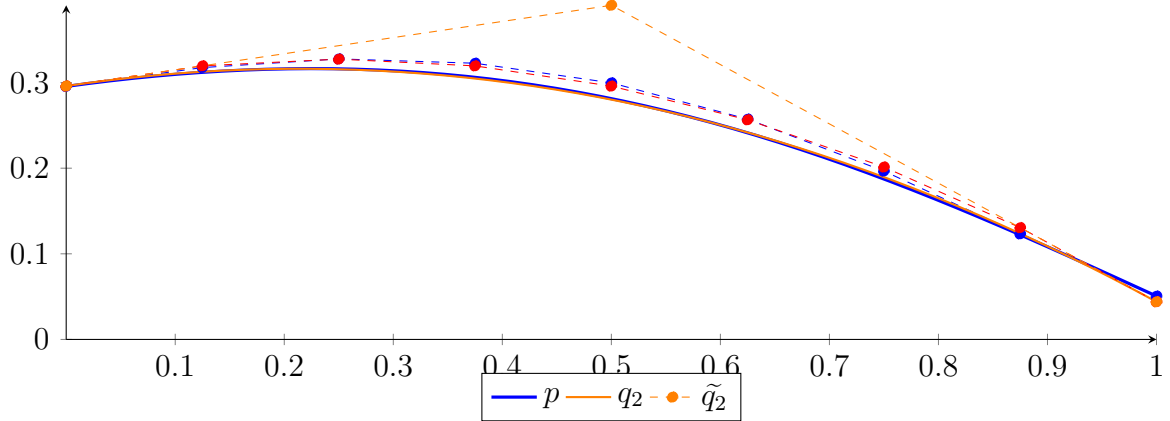
$$= 0.295967B_{0,2} + 0.390251B_{1,2} + 0.0439798B_{2,2}$$

$$\tilde{q}_2 = -5.14206 \cdot 10^{-13}X^8 + 2.08758 \cdot 10^{-12}X^7 - 3.46544 \cdot 10^{-12}X^6 + 3.01427 \cdot 10^{-12}X^5$$

$$- 1.45709 \cdot 10^{-12}X^4 + 3.78886 \cdot 10^{-13}X^3 - 0.440555X^2 + 0.188569X + 0.295967$$

$$= 0.295967B_{0,8} + 0.319538B_{1,8} + 0.327375B_{2,8} + 0.319477B_{3,8} + 0.295846B_{4,8}$$

$$+ 0.256481B_{5,8} + 0.201381B_{6,8} + 0.130548B_{7,8} + 0.0439798B_{8,8}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.00732213$ .

**Bounding polynomials  $M$  and  $m$ :**

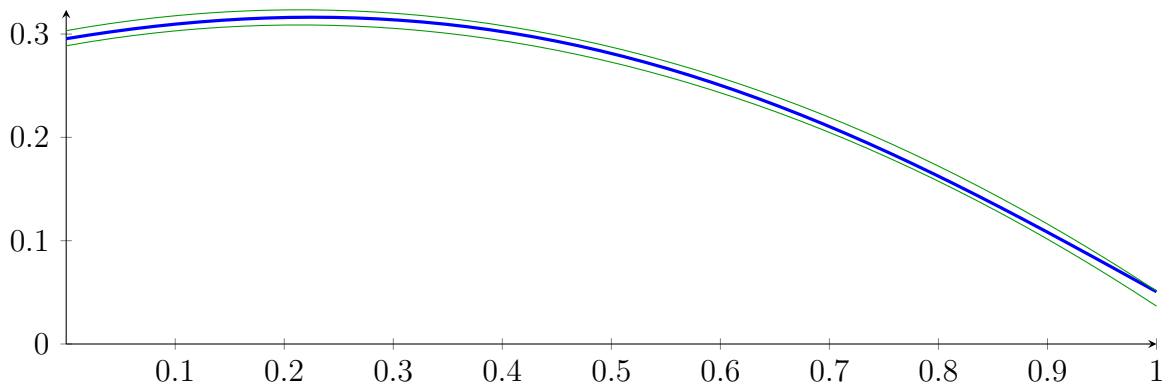
$$M = -0.440555X^2 + 0.188569X + 0.303289$$

$$m = -0.440555X^2 + 0.188569X + 0.288644$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{-0.642857, 1.07088\} \quad N(m) = \{-0.623236, 1.05126\}$$

**Intersection intervals:**

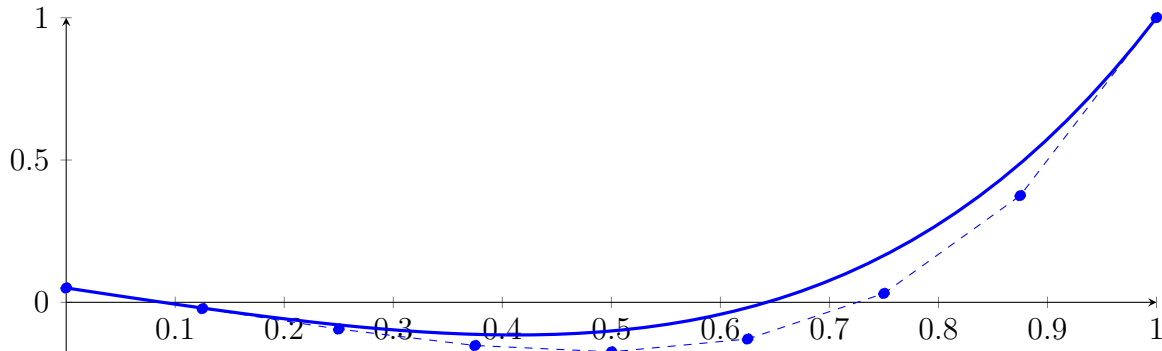


No intersection intervals with the  $x$  axis.

## 2.19 Recursion Branch 1 2 2 2 on the Second Half [0.875, 1]

Normalized monomial und Bézier representations and the Bézier polygon:

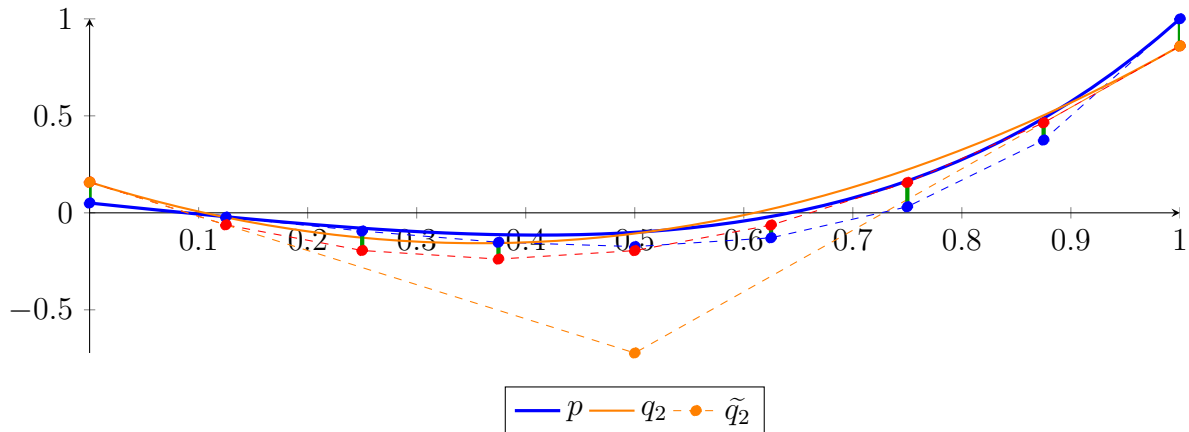
$$\begin{aligned}
 p &= 0.000126243X^8 + 0.00309849X^7 + 0.0320268X^6 + 0.178329X^5 \\
 &\quad + 0.544363X^4 + 0.75196X^3 + 0.0206513X^2 - 0.581138X + 0.0505832 \\
 &= 0.0505832B_{0,8}(X) - 0.022059B_{1,8}(X) - 0.0939636B_{2,8}(X) - 0.151703B_{3,8}(X) \\
 &\quad - 0.174072B_{4,8}(X) - 0.128906B_{5,8}(X) + 0.03125B_{6,8}(X) + 0.375B_{7,8}(X) + 1B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= 2.46306X^2 - 1.76145X + 0.158164 \\
 &= 0.158164B_{0,2} - 0.72256B_{1,2} + 0.859773B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -4.46706 \cdot 10^{-13}X^8 + 1.72173 \cdot 10^{-12}X^7 - 2.70413 \cdot 10^{-12}X^6 + 2.23122 \cdot 10^{-12}X^5 \\
 &\quad - 1.03962 \cdot 10^{-12}X^4 + 2.72651 \cdot 10^{-13}X^3 + 2.46306X^2 - 1.76145X + 0.158164 \\
 &= 0.158164B_{0,8} - 0.0620172B_{1,8} - 0.194232B_{2,8} - 0.23848B_{3,8} - 0.194762B_{4,8} \\
 &\quad - 0.0630779B_{5,8} + 0.156573B_{6,8} + 0.46419B_{7,8} + 0.859773B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.140227$ .

Bounding polynomials  $M$  and  $m$ :

$$M = 2.46306X^2 - 1.76145X + 0.298391$$

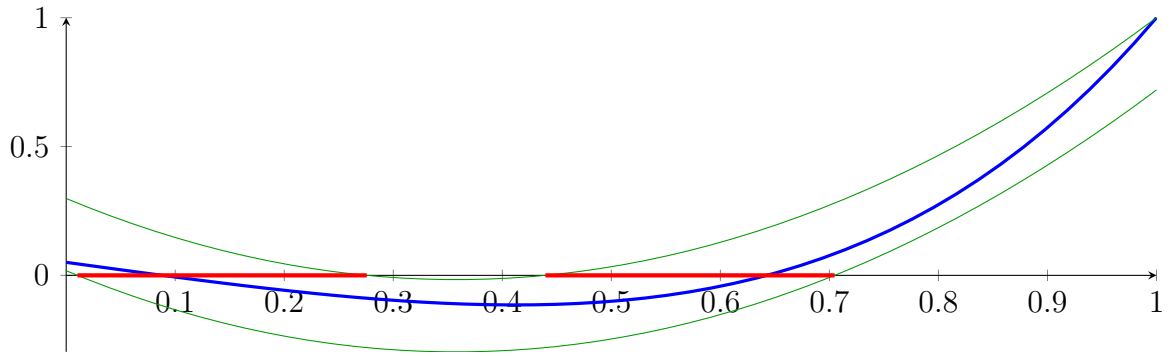
$$m = 2.46306X^2 - 1.76145X + 0.0179364$$

Root of  $M$  and  $m$ :

$$N(M) = \{0.275646, 0.439501\}$$

$$N(m) = \{0.010332, 0.704815\}$$

Intersection intervals:



$$[0.010332, 0.275646], [0.439501, 0.704815]$$

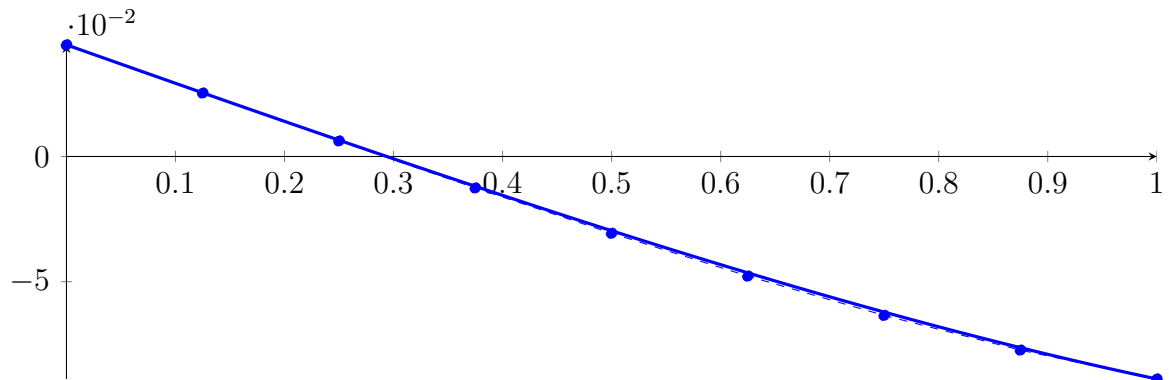
Longest intersection interval: 0.265314

⇒ Selective recursion: interval 1: [0.876292, 0.909456], interval 2: [0.929938, 0.963102],

## 2.20 Recursion Branch 1 2 2 2 1 in Interval 1: [0.876292, 0.909456]

Normalized monomial und Bézier representations and the Bézier polygon:

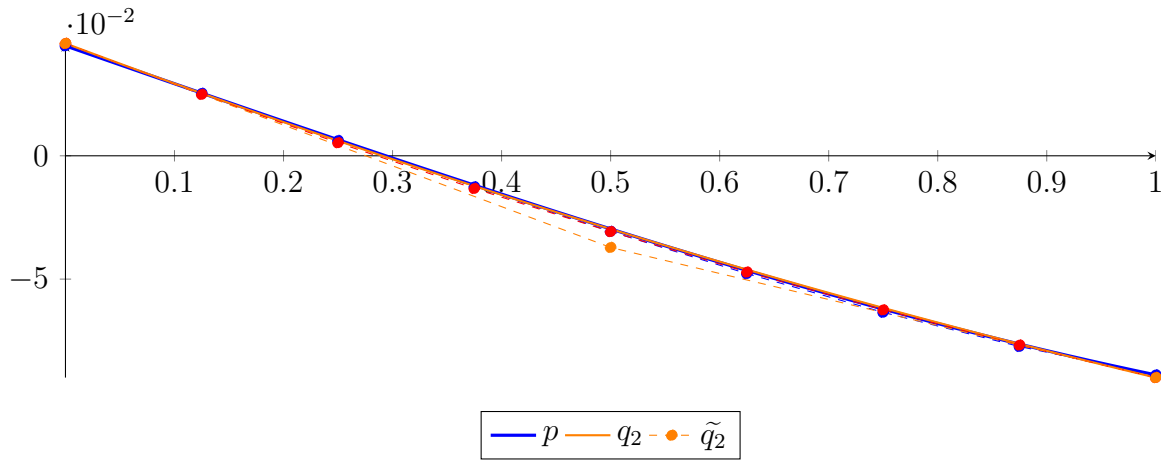
$$\begin{aligned} p &= 3.09944 \cdot 10^{-09} X^8 + 2.87692 \cdot 10^{-07} X^7 + 1.12487 \cdot 10^{-05} X^6 + 0.000237052 X^5 \\ &\quad + 0.00274319 X^4 + 0.0144672 X^3 + 0.00311902 X^2 - 0.154006 X + 0.044582 \\ &= 0.044582 B_{0,8}(X) + 0.0253312 B_{1,8}(X) + 0.00619186 B_{2,8}(X) \\ &\quad - 0.0125778 B_{3,8}(X) - 0.0306801 B_{4,8}(X) - 0.0477743 B_{5,8}(X) \\ &\quad - 0.0634712 B_{6,8}(X) - 0.0773287 B_{7,8}(X) - 0.0888461 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 0.0299663 X^2 - 0.165462 X + 0.0455672 \\ &= 0.0455672 B_{0,2} - 0.0371635 B_{1,2} - 0.089928 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 7.53218 \cdot 10^{-14} X^8 - 2.95004 \cdot 10^{-13} X^7 + 4.70452 \cdot 10^{-13} X^6 - 3.91764 \cdot 10^{-13} X^5 \\ &\quad + 1.80836 \cdot 10^{-13} X^4 - 4.47407 \cdot 10^{-14} X^3 + 0.0299663 X^2 - 0.165462 X + 0.0455672 \\ &= 0.0455672 B_{0,8} + 0.0248845 B_{1,8} + 0.00527206 B_{2,8} - 0.0132702 B_{3,8} \\ &\quad - 0.0307422 B_{4,8} - 0.047144 B_{5,8} - 0.0624756 B_{6,8} - 0.0767369 B_{7,8} - 0.089928 B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.00108196$ .

**Bounding polynomials  $M$  and  $m$ :**

$$M = 0.0299663X^2 - 0.165462X + 0.0466492$$

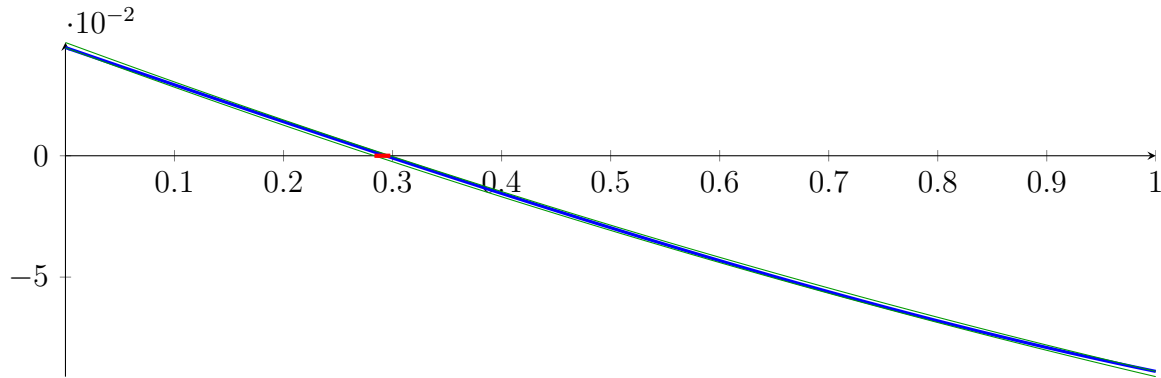
$$m = 0.0299663X^2 - 0.165462X + 0.0444853$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{0.298019, 5.22357\}$$

$$N(m) = \{0.283402, 5.23819\}$$

**Intersection intervals:**



$$[0.283402, 0.298019]$$

Longest intersection interval: 0.0146173

$\implies$  Selective recursion: interval 1:  $[0.88569, 0.886175]$ ,

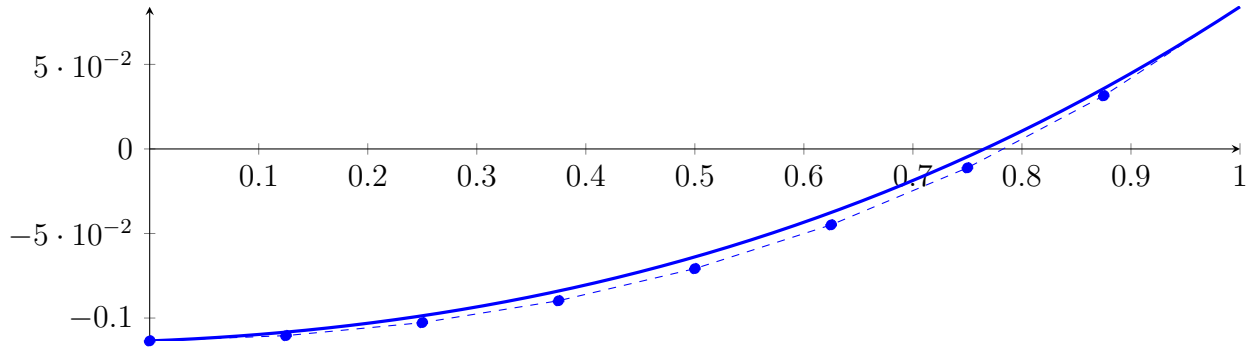
## 2.21 Recursion Branch 1 2 2 2 1 1 in Interval 1: $[0.88569, 0.886175]$

Found root in interval  $[0.88569, 0.886175]$  at recursion depth 6!

## 2.22 Recursion Branch 1 2 2 2 2 in Interval 2: $[0.929938, 0.963102]$

**Normalized monomial und Bézier representations and the Bézier polygon:**

$$\begin{aligned}
 p &= 3.09944 \cdot 10^{-09} X^8 + 3.27801 \cdot 10^{-07} X^7 + 1.47334 \cdot 10^{-05} X^6 + 0.000362771 X^5 \\
 &\quad + 0.00514607 X^4 + 0.0394424 X^3 + 0.127649 X^2 + 0.0249917 X - 0.113525 \\
 &= -0.113525 B_{0,8}(X) - 0.110401 B_{1,8}(X) - 0.102718 B_{2,8}(X) - 0.0897721 B_{3,8}(X) - 0.070785 B_{4,8}(X) \\
 &\quad - 0.0448989 B_{5,8}(X) - 0.0111691 B_{6,8}(X) + 0.0314439 B_{7,8}(X) + 0.0840821 B_{8,8}(X)
 \end{aligned}$$



**Degree reduction and raising:**

$$q_2 = 0.196309X^2 - 0.00378462X - 0.111071$$

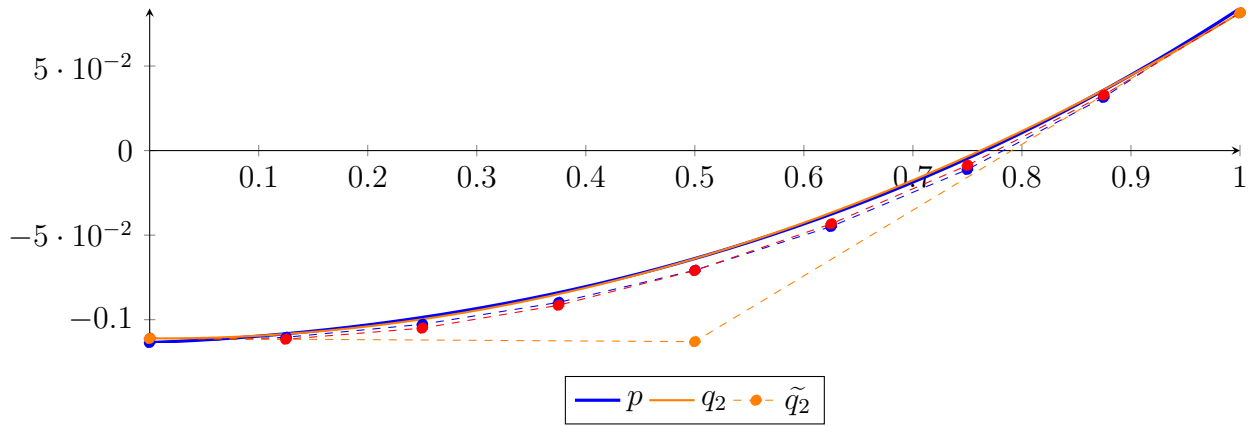
$$= -0.111071B_{0,2} - 0.112963B_{1,2} + 0.0814535B_{2,2}$$

$$\tilde{q}_2 = 7.91855 \cdot 10^{-14}X^8 - 3.32352 \cdot 10^{-13}X^7 + 5.70763 \cdot 10^{-13}X^6 - 5.13023 \cdot 10^{-13}X^5$$

$$+ 2.55365 \cdot 10^{-13}X^4 - 6.79055 \cdot 10^{-14}X^3 + 0.196309X^2 - 0.00378462X - 0.111071$$

$$= -0.111071B_{0,8} - 0.111544B_{1,8} - 0.105006B_{2,8} - 0.0914573B_{3,8} - 0.0708972B_{4,8}$$

$$- 0.0433261B_{5,8} - 0.00874394B_{6,8} + 0.0328493B_{7,8} + 0.0814535B_{8,8}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.00262863$ .

**Bounding polynomials  $M$  and  $m$ :**

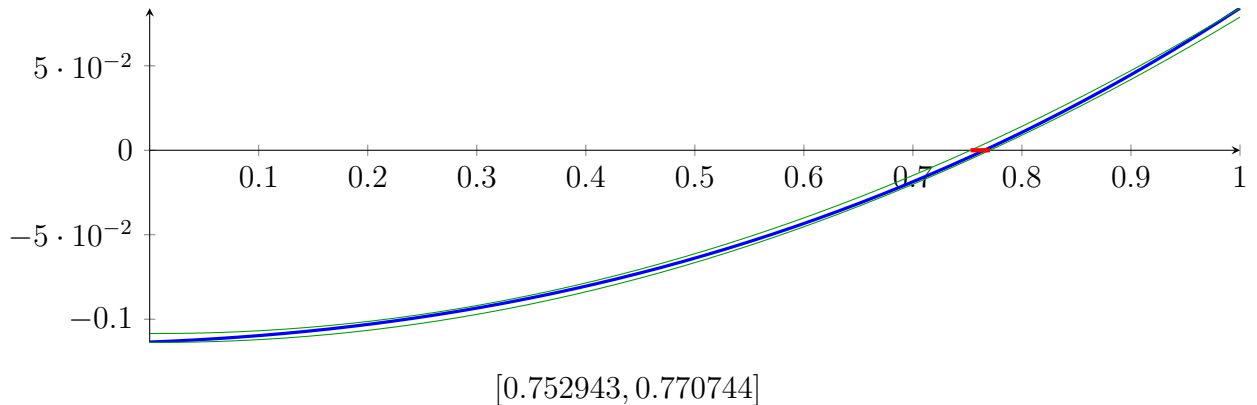
$$M = 0.196309X^2 - 0.00378462X - 0.108443$$

$$m = 0.196309X^2 - 0.00378462X - 0.1137$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{-0.733664, 0.752943\} \quad N(m) = \{-0.751465, 0.770744\}$$

**Intersection intervals:**



Longest intersection interval: 0.0178013

$\implies$  Selective recursion: interval 1:  $[0.954908, 0.955499]$ ,

## **2.23 Recursion Branch 1 2 2 2 2 1 in Interval 1: [0.954908, 0.955499]**

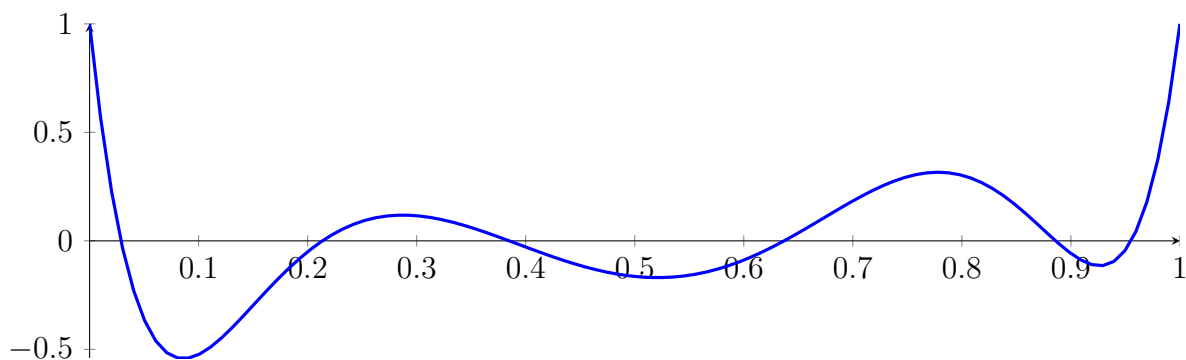
Found root in interval [0.954908, 0.955499] at recursion depth 6!



## 2.24 Result: 6 Root Intervals

Input Polynomial on Interval  $[0, 1]$

$$p = 2118X^8 - 8328X^7 + 14000X^6 - 13216X^5 + 7630X^4 - 2688X^3 + 532X^2 - 48X + 1$$



**Result: Root Intervals**

$$[0.0287798, 0.0287982], [0.213531, 0.213532], [0.384513, 0.385108], [0.636926, 0.637038], \\ [0.88569, 0.886175], [0.954908, 0.955499]$$

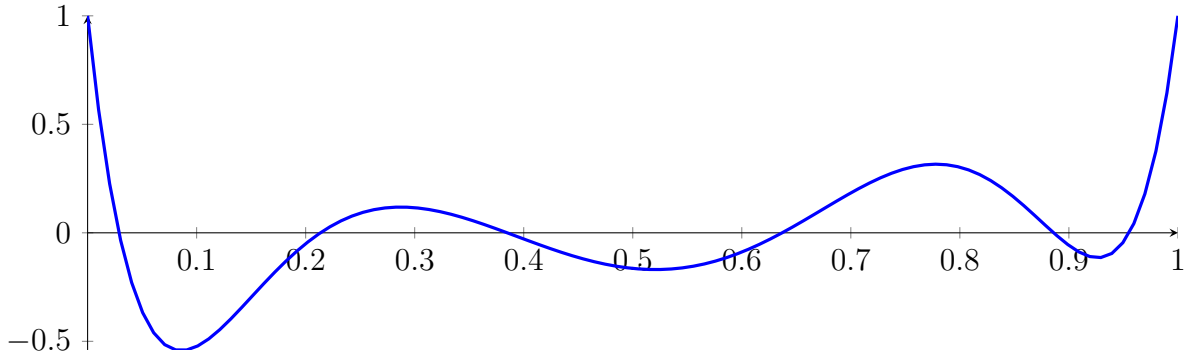
with precision  $\varepsilon = 0.001$ .

### 3 CubeClip Applied to a Polynomial of 8th Degree with Six Roots

$$2118X^8 - 8328X^7 + 14000X^6 - 13216X^5 + 7630X^4 - 2688X^3 + 532X^2 - 48X + 1$$

Called CubeClip with input polynomial on interval  $[0, 1]$ :

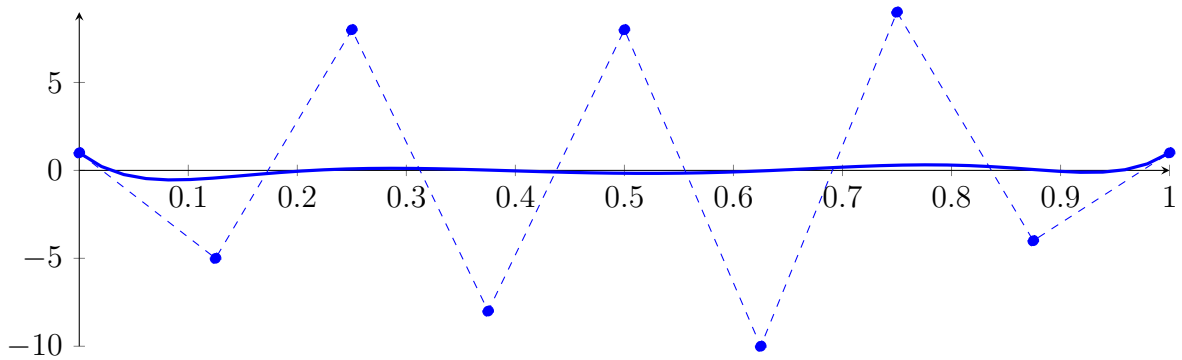
$$p = 2118X^8 - 8328X^7 + 14000X^6 - 13216X^5 + 7630X^4 - 2688X^3 + 532X^2 - 48X + 1$$



#### 3.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

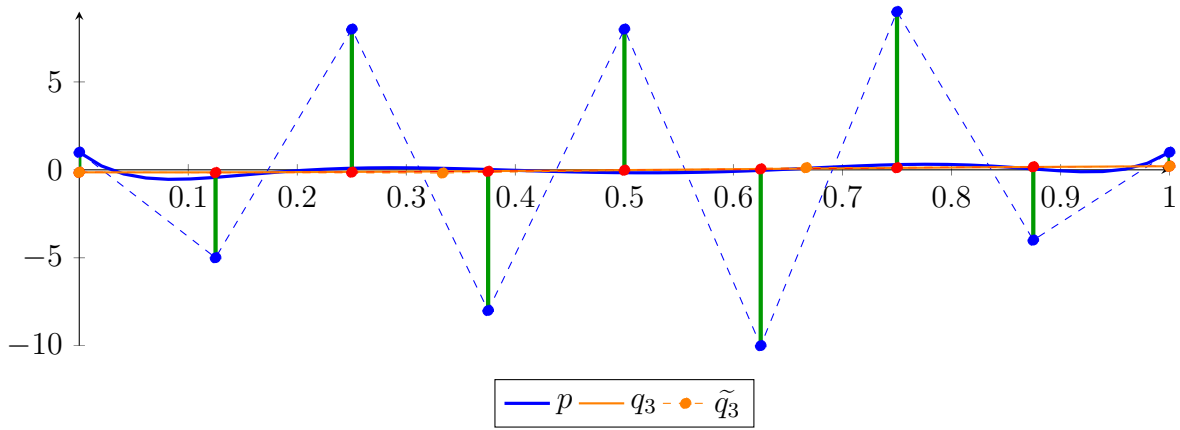
$$\begin{aligned} p &= 2118X^8 - 8328X^7 + 14000X^6 - 13216X^5 + 7630X^4 - 2688X^3 + 532X^2 - 48X + 1 \\ &= 1B_{0,8}(X) - 5B_{1,8}(X) + 8B_{2,8}(X) - 8B_{3,8}(X) + 8B_{4,8}(X) \\ &\quad - 10B_{5,8}(X) + 9B_{6,8}(X) - 4B_{7,8}(X) + 1B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.565657X^3 + 1.0303X^2 - 0.121212X - 0.141414 \\ &= -0.141414B_{0,3} - 0.181818B_{1,3} + 0.121212B_{2,3} + 0.20202B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 1.85402 \cdot 10^{-13}X^8 - 7.54964 \cdot 10^{-13}X^7 + 1.25787 \cdot 10^{-12}X^6 - 1.10041 \cdot 10^{-12}X^5 \\ &\quad + 5.38663 \cdot 10^{-13}X^4 - 0.565657X^3 + 1.0303X^2 - 0.121212X - 0.141414 \\ &= -0.141414B_{0,8} - 0.156566B_{1,8} - 0.134921B_{2,8} - 0.0865801B_{3,8} \\ &\quad - 0.021645B_{4,8} + 0.0497835B_{5,8} + 0.117605B_{6,8} + 0.171717B_{7,8} + 0.20202B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 10.0498$ .

**Bounding polynomials  $M$  and  $m$ :**

$$M = -0.565657X^3 + 1.0303X^2 - 0.121212X + 9.90837$$

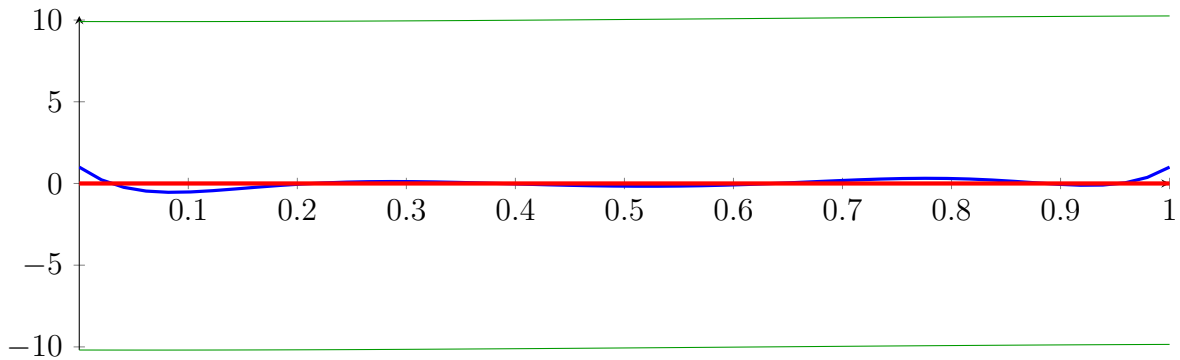
$$m = -0.565657X^3 + 1.0303X^2 - 0.121212X - 10.1912$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{3.33349\}$$

$$N(m) = \{-2.11288\}$$

**Intersection intervals:**



$[0, 1]$

Longest intersection interval: 1

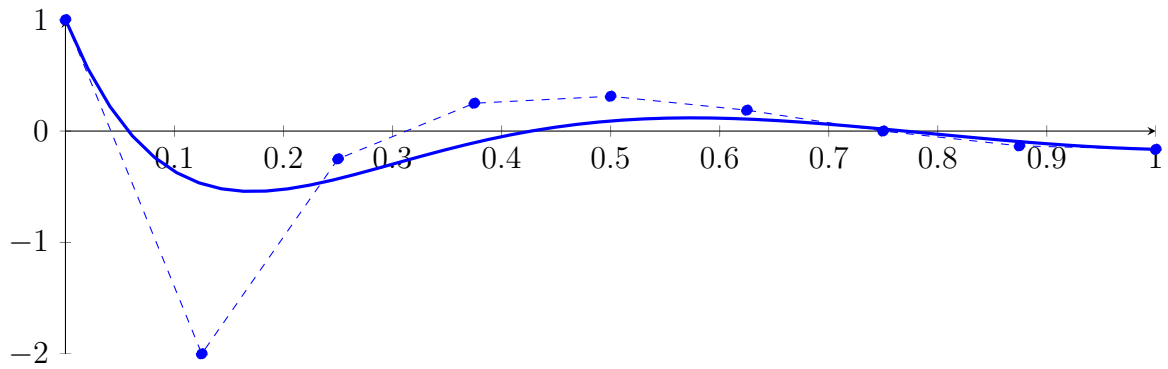
$\implies$  Bisection: first half  $[0, 0.5]$  und second half  $[0.5, 1]$

Bisection point is very near to a root?!?

### 3.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

**Normalized monomial und Bézier representations and the Bézier polygon:**

$$\begin{aligned} p &= 8.27344X^8 - 65.0625X^7 + 218.75X^6 - 413X^5 + 476.875X^4 - 336X^3 + 133X^2 - 24X + 1 \\ &= 1B_{0,8}(X) - 2B_{1,8}(X) - 0.25B_{2,8}(X) + 0.25B_{3,8}(X) + 0.3125B_{4,8}(X) \\ &\quad + 0.1875B_{5,8}(X) - 2.66443 \cdot 10^{-17}B_{6,8}(X) - 0.132813B_{7,8}(X) - 0.164063B_{8,8}(X) \end{aligned}$$



**Degree reduction and raising:**

$$q_3 = -9.06597X^3 + 13.4044X^2 - 5.06061X + 0.240136$$

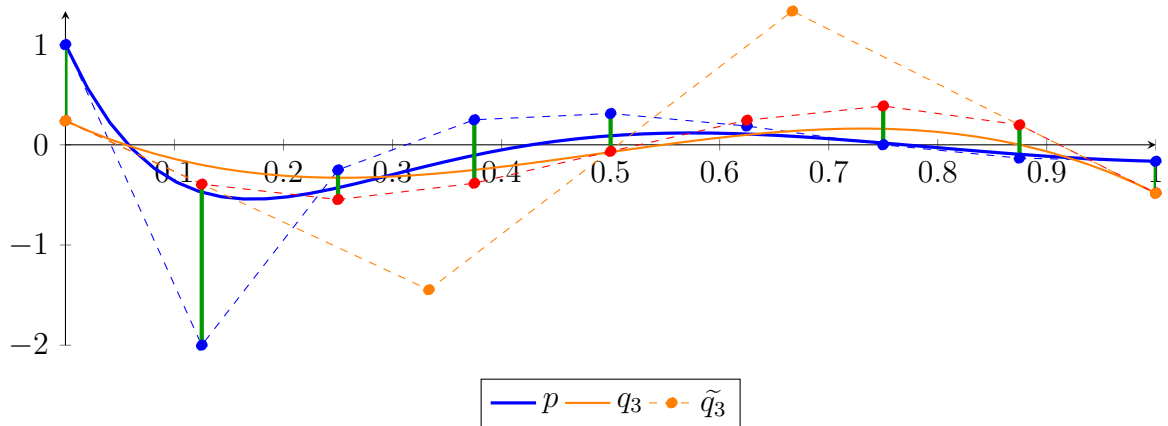
$$= 0.240136B_{0,3} - 1.44673B_{1,3} + 1.33452B_{2,3} - 0.482086B_{3,3}$$

$$\tilde{q}_3 = 5.72775 \cdot 10^{-13}X^8 - 2.5956 \cdot 10^{-12}X^7 + 4.80821 \cdot 10^{-12}X^6 - 4.67776 \cdot 10^{-12}X^5$$

$$+ 2.55068 \cdot 10^{-12}X^4 - 9.06597X^3 + 13.4044X^2 - 5.06061X + 0.240136$$

$$= 0.240136B_{0,8} - 0.39244B_{1,8} - 0.546289B_{2,8} - 0.383303B_{3,8} - 0.0653747B_{4,8}$$

$$+ 0.245603B_{5,8} + 0.387739B_{6,8} + 0.19914B_{7,8} - 0.482086B_{8,8}$$



The maximum difference of the Bézier coefficients is  $\delta = 1.60756$ .

**Bounding polynomials  $M$  and  $m$ :**

$$M = -9.06597X^3 + 13.4044X^2 - 5.06061X + 1.8477$$

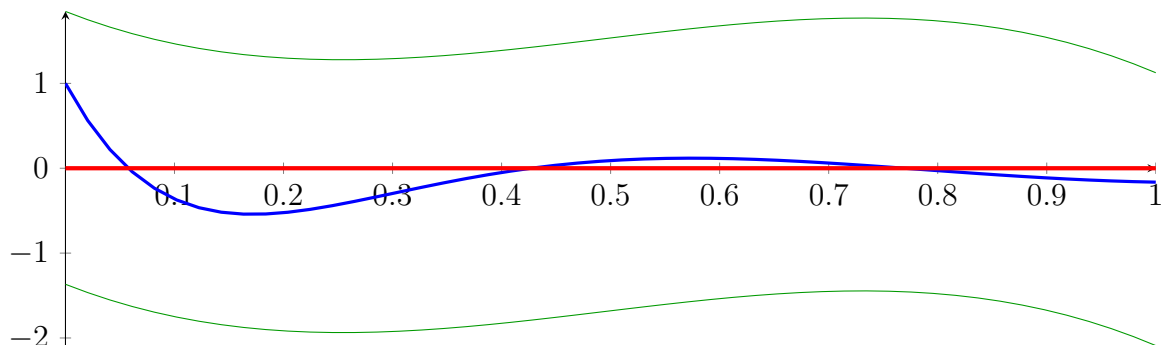
$$m = -9.06597X^3 + 13.4044X^2 - 5.06061X - 1.36742$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{1.14675\}$$

$$N(m) = \{-0.177139\}$$

**Intersection intervals:**



[0, 1]

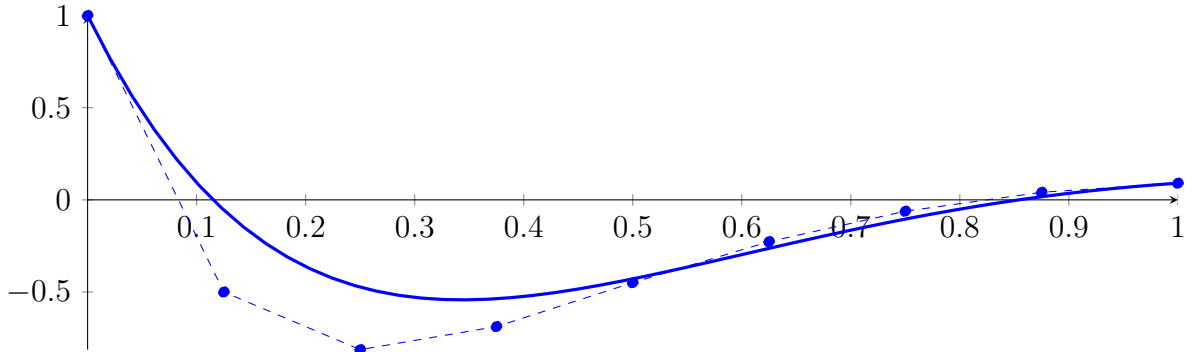
Longest intersection interval: 1

⇒ Bisection: first half [0, 0.25] und second half [0.25, 0.5]

### 3.3 Recursion Branch 1 1 1 on the First Half [0, 0.25]

Normalized monomial und Bézier representations and the Bézier polygon:

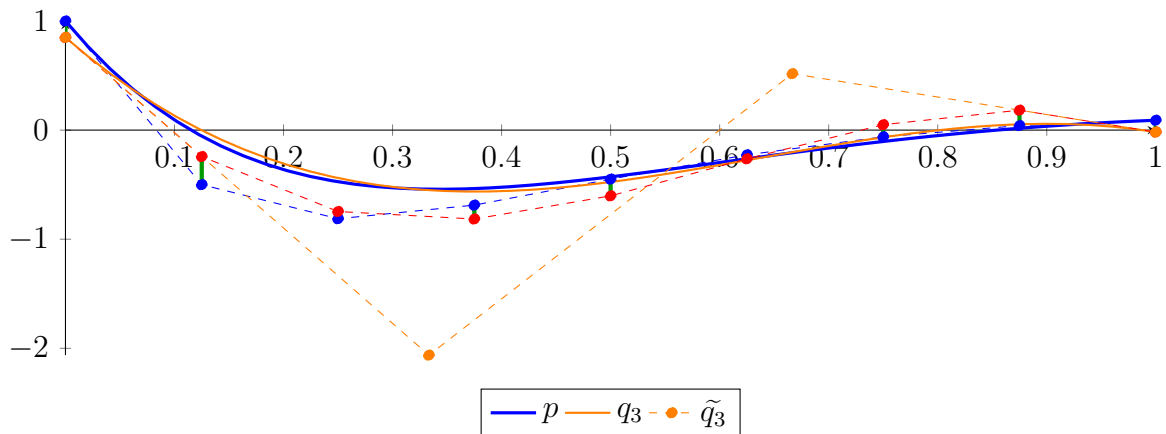
$$\begin{aligned}
 p &= 0.0323181X^8 - 0.508301X^7 + 3.41797X^6 - 12.9063X^5 + 29.8047X^4 - 42X^3 + 33.25X^2 - 12X + 1 \\
 &= 1B_{0,8}(X) - 0.5B_{1,8}(X) - 0.8125B_{2,8}(X) - 0.6875B_{3,8}(X) - 0.449219B_{4,8}(X) \\
 &\quad - 0.226563B_{5,8}(X) - 0.0615234B_{6,8}(X) + 0.0409546B_{7,8}(X) + 0.0904236B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -8.607X^3 + 16.4762X^2 - 8.73691X + 0.849702 \\
 &= 0.849702B_{0,3} - 2.0626B_{1,3} + 0.517161B_{2,3} - 0.0180051B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 3.0155 \cdot 10^{-12}X^8 - 1.23387 \cdot 10^{-11}X^7 + 2.06667 \cdot 10^{-11}X^6 - 1.82054 \cdot 10^{-11}X^5 \\
 &\quad + 9.00471 \cdot 10^{-12}X^4 - 8.607X^3 + 16.4762X^2 - 8.73691X + 0.849702 \\
 &= 0.849702B_{0,8} - 0.242413B_{1,8} - 0.746091B_{2,8} - 0.81503B_{3,8} - 0.602925B_{4,8} \\
 &\quad - 0.263474B_{5,8} + 0.049627B_{6,8} + 0.182682B_{7,8} - 0.0180051B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.257587$ .

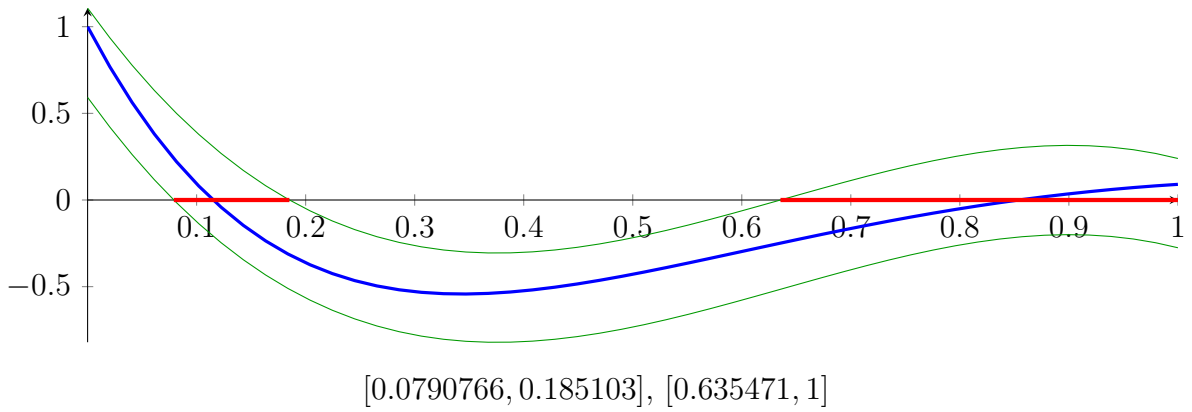
Bounding polynomials  $M$  and  $m$ :

$$\begin{aligned}
 M &= -8.607X^3 + 16.4762X^2 - 8.73691X + 1.10729 \\
 m &= -8.607X^3 + 16.4762X^2 - 8.73691X + 0.592114
 \end{aligned}$$

Root of  $M$  and  $m$ :

$$N(M) = \{0.185103, 0.635471, 1.09371\} \qquad N(m) = \{0.0790766\}$$

Intersection intervals:



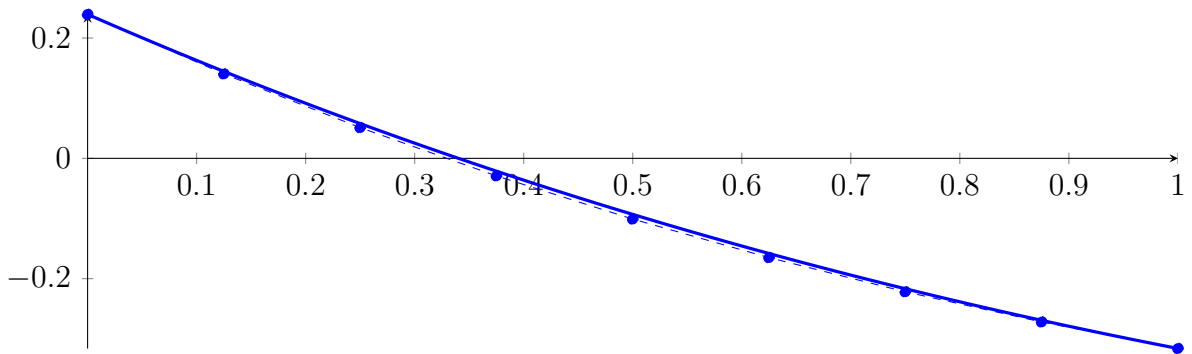
Longest intersection interval: 0.364529

⇒ Selective recursion: interval 1: [0.0197692, 0.0462757], interval 2: [0.158868, 0.25],

### 3.4 Recursion Branch 1 1 1 1 in Interval 1: [0.0197692, 0.0462757]

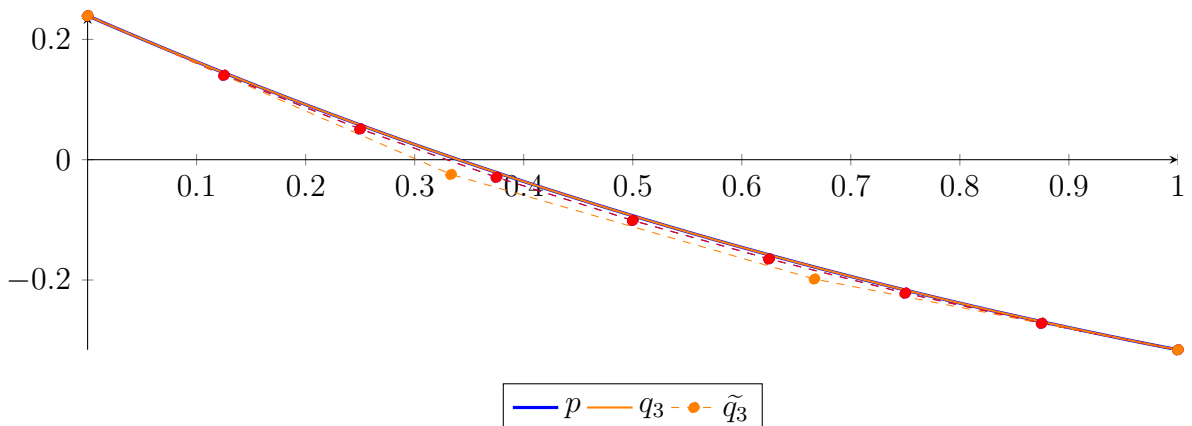
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 5.16117 \cdot 10^{-10} X^8 - 7.3482 \cdot 10^{-08} X^7 + 4.46395 \cdot 10^{-06} X^6 - 0.000152081 X^5 \\
 &\quad + 0.00316103 X^4 - 0.0397456 X^3 + 0.27365 X^2 - 0.792312 X + 0.239355 \\
 &= 0.239355 B_{0,8}(X) + 0.140316 B_{1,8}(X) + 0.0510498 B_{2,8}(X) - 0.0291526 B_{3,8}(X) \\
 &\quad - 0.100956 B_{4,8}(X) - 0.164983 B_{5,8}(X) - 0.221815 B_{6,8}(X) - 0.271998 B_{7,8}(X) - 0.31604 B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -0.0338314 X^3 + 0.269933 X^2 - 0.791496 X + 0.239314 \\
 &= 0.239314 B_{0,3} - 0.0245178 B_{1,3} - 0.198372 B_{2,3} - 0.31608 B_{3,3} \\
 \tilde{q}_3 &= 1.86093 \cdot 10^{-13} X^8 - 7.53843 \cdot 10^{-13} X^7 + 1.24812 \cdot 10^{-12} X^6 - 1.08545 \cdot 10^{-12} X^5 \\
 &\quad + 5.28903 \cdot 10^{-13} X^4 - 0.0338314 X^3 + 0.269933 X^2 - 0.791496 X + 0.239314 \\
 &= 0.239314 B_{0,8} + 0.140377 B_{1,8} + 0.0510807 B_{2,8} - 0.0291795 B_{3,8} \\
 &\quad - 0.101007 B_{4,8} - 0.165007 B_{5,8} - 0.221783 B_{6,8} - 0.271939 B_{7,8} - 0.31608 B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 6.15092 \cdot 10^{-05}$ .

**Bounding polynomials  $M$  and  $m$ :**

$$M = -0.0338314X^3 + 0.269933X^2 - 0.791496X + 0.239376$$

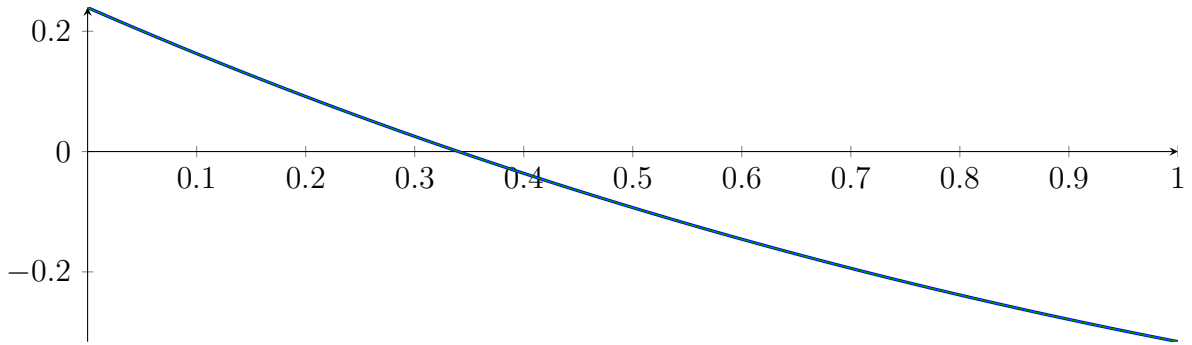
$$m = -0.0338314X^3 + 0.269933X^2 - 0.791496X + 0.239253$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{0.340229\}$$

$$N(m) = \{0.34003\}$$

**Intersection intervals:**



$$[0.34003, 0.340229]$$

Longest intersection interval: 0.000198541

⇒ Selective recursion: [interval 1: \[0.0287822, 0.0287874\]](#),

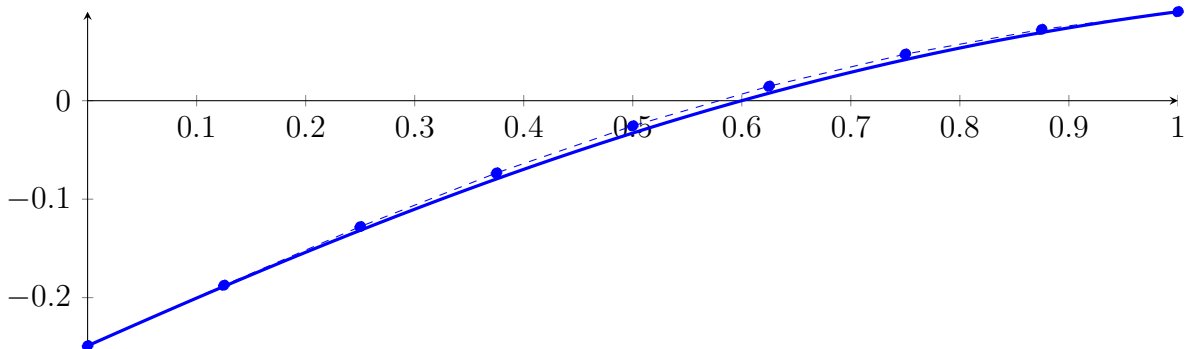
### 3.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.0287822, 0.0287874]

Found root in interval [0.0287822, 0.0287874] at recursion depth 5!

### 3.6 Recursion Branch 1 1 1 2 in Interval 2: [0.158868, 0.25]

**Normalized monomial und Bézier representations and the Bézier polygon:**

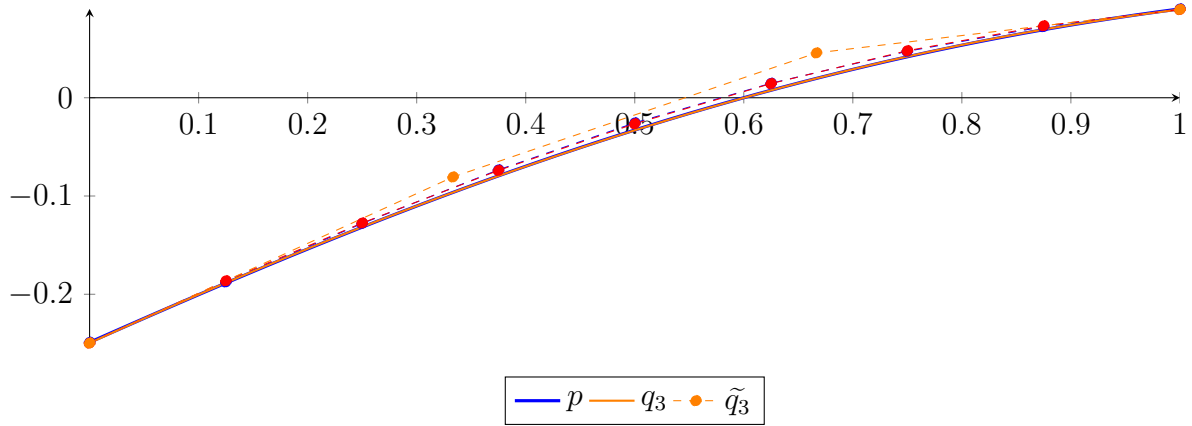
$$\begin{aligned} p &= 1.00763 \cdot 10^{-05} X^8 - 0.00029423 X^7 + 0.00357189 X^6 - 0.0239459 X^5 \\ &\quad + 0.0936609 X^4 - 0.170978 X^3 - 0.054483 X^2 + 0.491797 X - 0.248915 \\ &= -0.248915 B_{0,8}(X) - 0.187441 B_{1,8}(X) - 0.127912 B_{2,8}(X) - 0.0733819 B_{3,8}(X) - 0.0255662 B_{4,8}(X) \\ &\quad + 0.0147305 B_{5,8}(X) + 0.0473139 B_{6,8}(X) + 0.0723907 B_{7,8}(X) + 0.0904236 B_{8,8}(X) \end{aligned}$$



### Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.0393188X^3 - 0.128291X^2 + 0.507066X - 0.249646 \\ &= -0.249646B_{0,3} - 0.0806241B_{1,3} + 0.0456344B_{2,3} + 0.0898105B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.47113 \cdot 10^{-14}X^8 + 9.8569 \cdot 10^{-14}X^7 - 1.61082 \cdot 10^{-13}X^6 + 1.41159 \cdot 10^{-13}X^5 \\ &\quad - 7.35215 \cdot 10^{-14}X^4 - 0.0393188X^3 - 0.128291X^2 + 0.507066X - 0.249646 \\ &= -0.249646B_{0,8} - 0.186263B_{1,8} - 0.127461B_{2,8} - 0.0739439B_{3,8} - 0.0264124B_{4,8} \\ &\quad + 0.0144309B_{5,8} + 0.0478839B_{6,8} + 0.0732444B_{7,8} + 0.0898105B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.00117777$ .

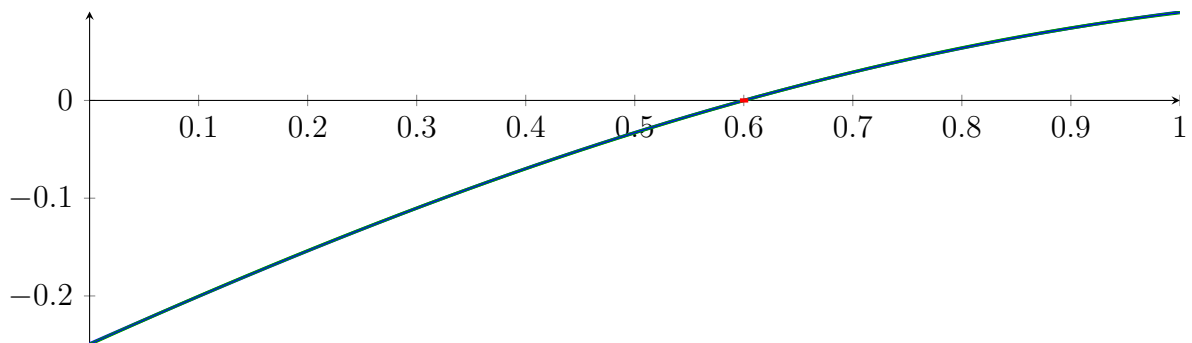
### Bounding polynomials $M$ and $m$ :

$$\begin{aligned} M &= -0.0393188X^3 - 0.128291X^2 + 0.507066X - 0.248468 \\ m &= -0.0393188X^3 - 0.128291X^2 + 0.507066X - 0.250824 \end{aligned}$$

### Root of $M$ and $m$ :

$$N(M) = \{-5.71355, 0.596487, 1.85423\} \quad N(m) = \{-5.7148, 0.604072, 1.8479\}$$

### Intersection intervals:



$$[0.596487, 0.604072]$$

Longest intersection interval: 0.00758528

$\implies$  Selective recursion: interval 1:  $[0.213227, 0.213918]$ ,

### 3.7 Recursion Branch 1 1 1 2 1 in Interval 1: $[0.213227, 0.213918]$

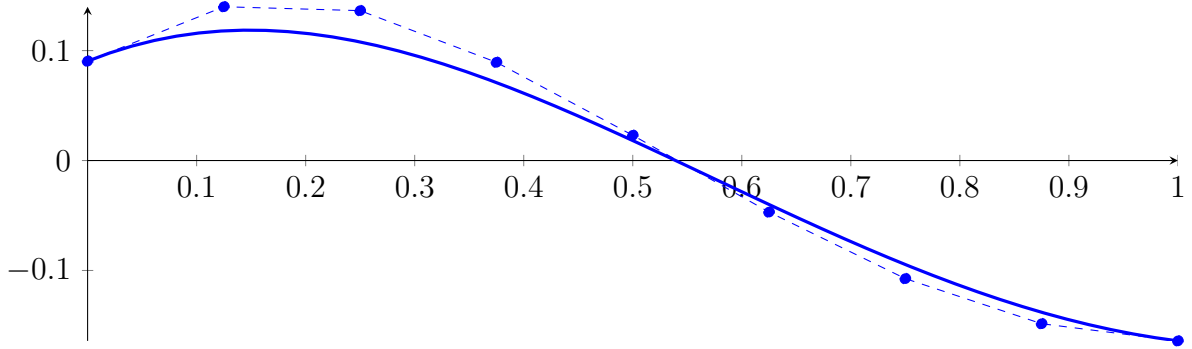
Found root in interval  $[0.213227, 0.213918]$  at recursion depth 5!



### 3.8 Recursion Branch 1 1 2 on the Second Half [0.25, 0.5]

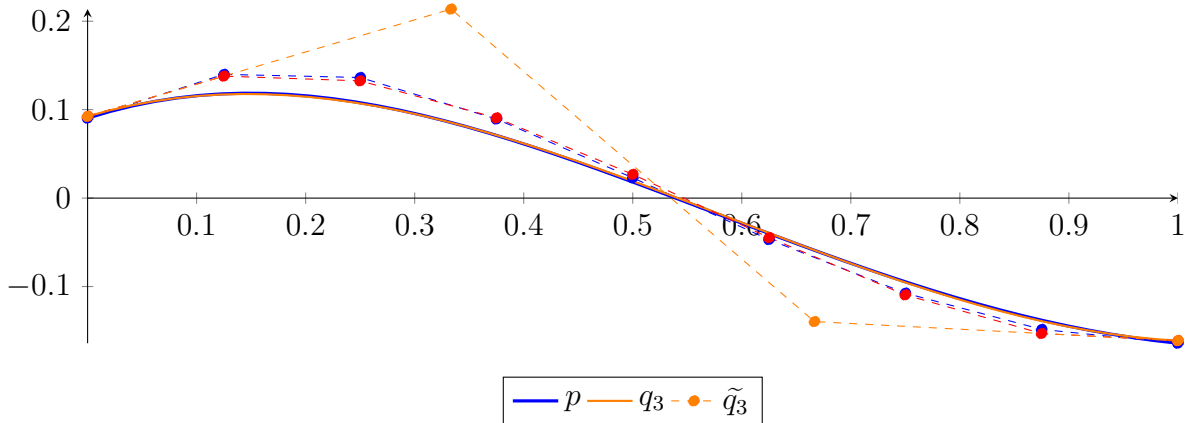
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 0.0323181X^8 - 0.249756X^7 + 0.764771X^6 - 1.26294X^5 \\
 &\quad + 1.01471X^4 + 0.534912X^3 - 1.48425X^2 + 0.395752X + 0.0904236 \\
 &= 0.0904236B_{0,8}(X) + 0.139893B_{1,8}(X) + 0.136353B_{2,8}(X) + 0.0893555B_{3,8}(X) + 0.0229492B_{4,8}(X) \\
 &\quad - 0.046875B_{5,8}(X) - 0.107422B_{6,8}(X) - 0.148438B_{7,8}(X) - 0.164063B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= 0.806244X^3 - 1.42357X^2 + 0.363718X + 0.092455 \\
 &= 0.092455B_{0,3} + 0.213694B_{1,3} - 0.139588B_{2,3} - 0.161149B_{3,3} \\
 \tilde{q}_3 &= -2.16878 \cdot 10^{-13}X^8 + 8.8954 \cdot 10^{-13}X^7 - 1.49388 \cdot 10^{-12}X^6 + 1.31884 \cdot 10^{-12}X^5 \\
 &\quad - 6.52769 \cdot 10^{-13}X^4 + 0.806244X^3 - 1.42357X^2 + 0.363718X + 0.092455 \\
 &= 0.092455B_{0,8} + 0.13792B_{1,8} + 0.132543B_{2,8} + 0.0907215B_{3,8} + 0.026853B_{4,8} \\
 &\quad - 0.0446656B_{5,8} - 0.109437B_{6,8} - 0.153064B_{7,8} - 0.161149B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.00462634$ .

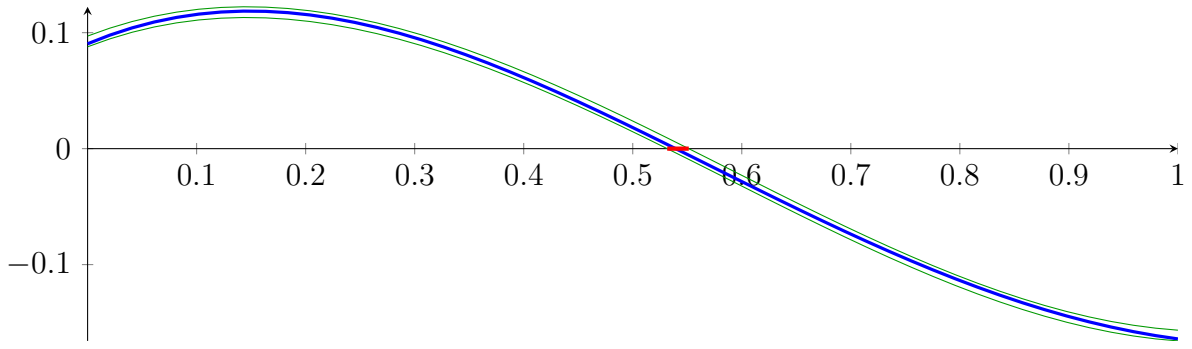
Bounding polynomials  $M$  and  $m$ :

$$\begin{aligned}
 M &= 0.806244X^3 - 1.42357X^2 + 0.363718X + 0.0970813 \\
 m &= 0.806244X^3 - 1.42357X^2 + 0.363718X + 0.0878287
 \end{aligned}$$

Root of  $M$  and  $m$ :

$$N(M) = \{-0.159023, 0.551352, 1.37335\} \quad N(m) = \{-0.148242, 0.531611, 1.38231\}$$

Intersection intervals:



[0.531611, 0.551352]

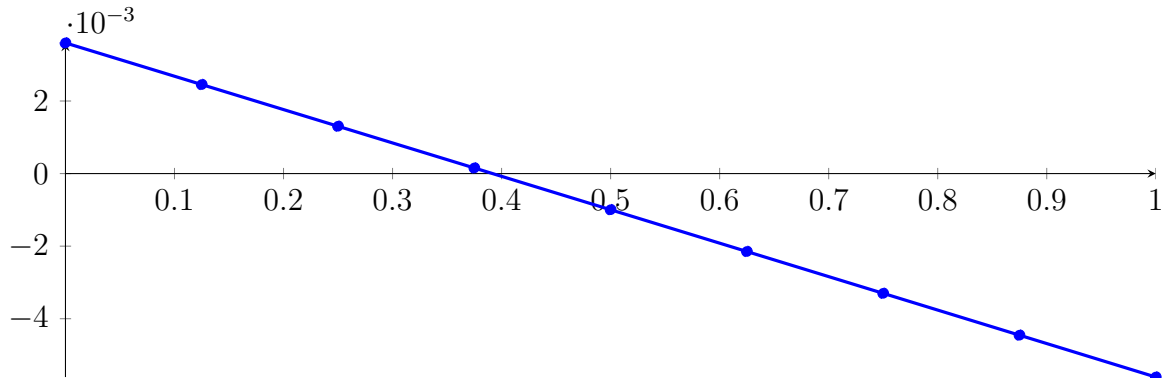
Longest intersection interval: 0.0197414

⇒ Selective recursion: interval 1: [0.382903, 0.387838],

### 3.9 Recursion Branch 1 1 2 1 in Interval 1: [0.382903, 0.387838]

Normalized monomial und Bézier representations and the Bézier polygon:

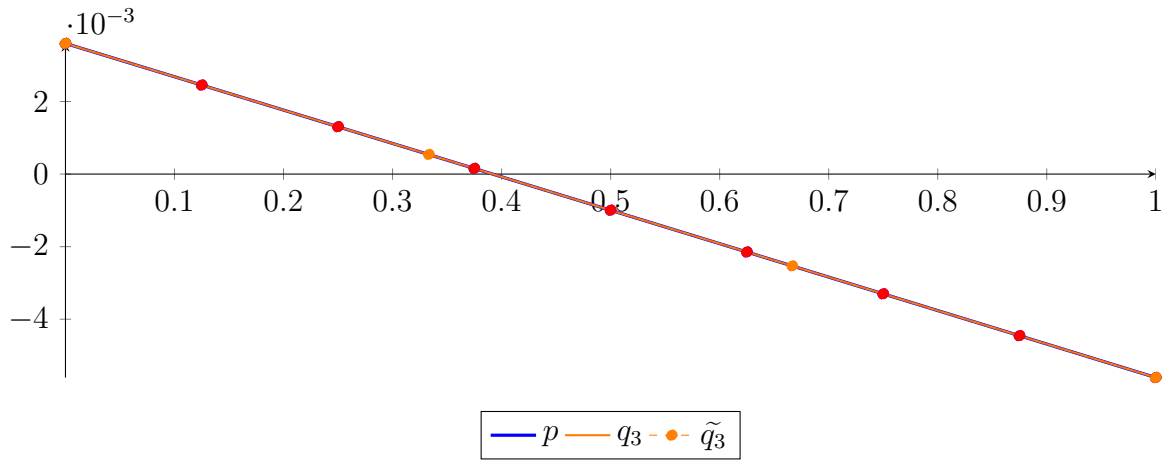
$$\begin{aligned}
 p &= 7.45613 \cdot 10^{-16} X^8 - 1.31238 \cdot 10^{-13} X^7 + 5.39216 \cdot 10^{-12} X^6 - 1.01729 \cdot 10^{-10} X^5 - 3.53726 \\
 &\quad \cdot 10^{-08} X^4 + 6.1555 \cdot 10^{-06} X^3 - 3.6645 \cdot 10^{-05} X^2 - 0.00917787 X + 0.0036009 \\
 &= 0.0036009 B_{0,8}(X) + 0.00245366 B_{1,8}(X) + 0.00130512 B_{2,8}(X) \\
 &\quad + 0.00015538 B_{3,8}(X) - 0.000995451 B_{4,8}(X) - 0.00214726 B_{5,8}(X) \\
 &\quad - 0.00329995 B_{6,8}(X) - 0.00445339 B_{7,8}(X) - 0.0056075 B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= 6.08449 \cdot 10^{-06} X^3 - 3.65993 \cdot 10^{-05} X^2 - 0.00917788 X + 0.0036009 \\
 &= 0.0036009 B_{0,3} + 0.000541605 B_{1,3} - 0.00252989 B_{2,3} - 0.0056075 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 4.26679 \cdot 10^{-16} X^8 - 1.85342 \cdot 10^{-15} X^7 + 3.29239 \cdot 10^{-15} X^6 - 3.08497 \cdot 10^{-15} X^5 + 1.62513 \\
 &\quad \cdot 10^{-15} X^4 + 6.08449 \cdot 10^{-06} X^3 - 3.65993 \cdot 10^{-05} X^2 - 0.00917788 X + 0.0036009 \\
 &= 0.0036009 B_{0,8} + 0.00245366 B_{1,8} + 0.00130512 B_{2,8} + 0.00015538 B_{3,8} - 0.000995451 B_{4,8} \\
 &\quad - 0.00214726 B_{5,8} - 0.00329995 B_{6,8} - 0.0044534 B_{7,8} - 0.0056075 B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 7.63896 \cdot 10^{-10}$ .

**Bounding polynomials  $M$  and  $m$ :**

$$M = 6.08449 \cdot 10^{-06} X^3 - 3.65993 \cdot 10^{-05} X^2 - 0.00917788 X + 0.0036009$$

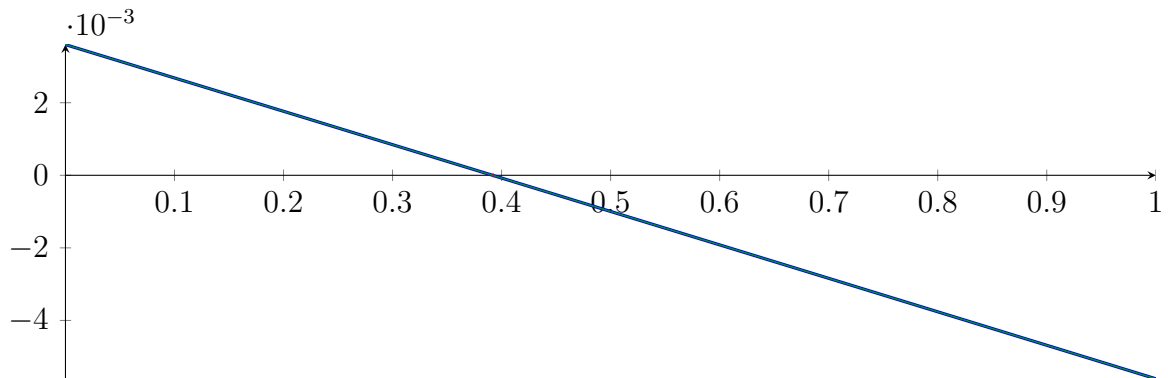
$$m = 6.08449 \cdot 10^{-06} X^3 - 3.65993 \cdot 10^{-05} X^2 - 0.00917788 X + 0.0036009$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{-36.1564, 0.391773, 41.7798\}$$

$$N(m) = \{-36.1564, 0.391773, 41.7798\}$$

**Intersection intervals:**



$$[0.391773, 0.391773]$$

Longest intersection interval:  $1.65997 \cdot 10^{-07}$

$\implies$  Selective recursion: interval 1:  $[0.384836, 0.384836]$ ,

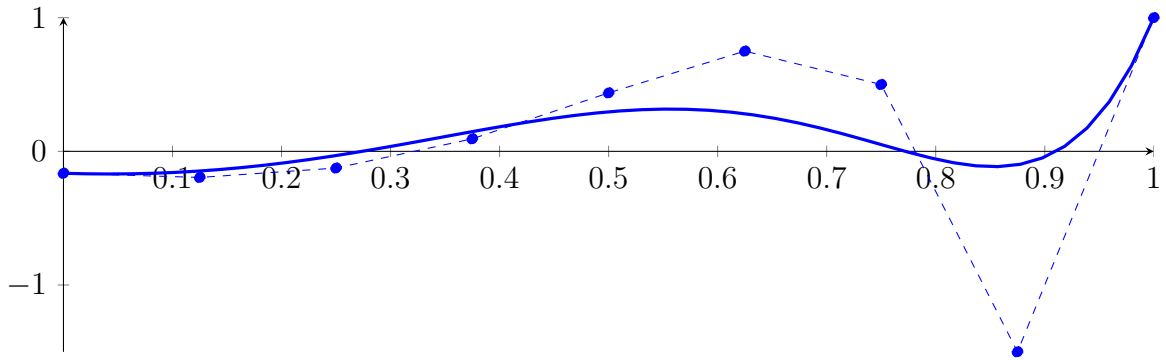
### 3.10 Recursion Branch 1 1 2 1 1 in Interval 1: $[0.384836, 0.384836]$

Found root in interval  $[0.384836, 0.384836]$  at recursion depth 5!

### 3.11 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

**Normalized monomial und Bézier representations and the Bézier polygon:**

$$\begin{aligned} p &= 8.27344X^8 + 1.125X^7 - 5.03125X^6 - 3.5X^5 - 4.92188X^4 + 2.625X^3 + 2.84375X^2 - 0.25X - 0.164063 \\ &= -0.164063B_{0,8}(X) - 0.195313B_{1,8}(X) - 0.125B_{2,8}(X) + 0.09375B_{3,8}(X) \\ &\quad + 0.4375B_{4,8}(X) + 0.75B_{5,8}(X) + 0.5B_{6,8}(X) - 1.5B_{7,8}(X) + 1B_{8,8}(X) \end{aligned}$$



**Degree reduction and raising:**

$$q_3 = 3.2238X^3 - 5.7178X^2 + 3.16951X - 0.396228$$

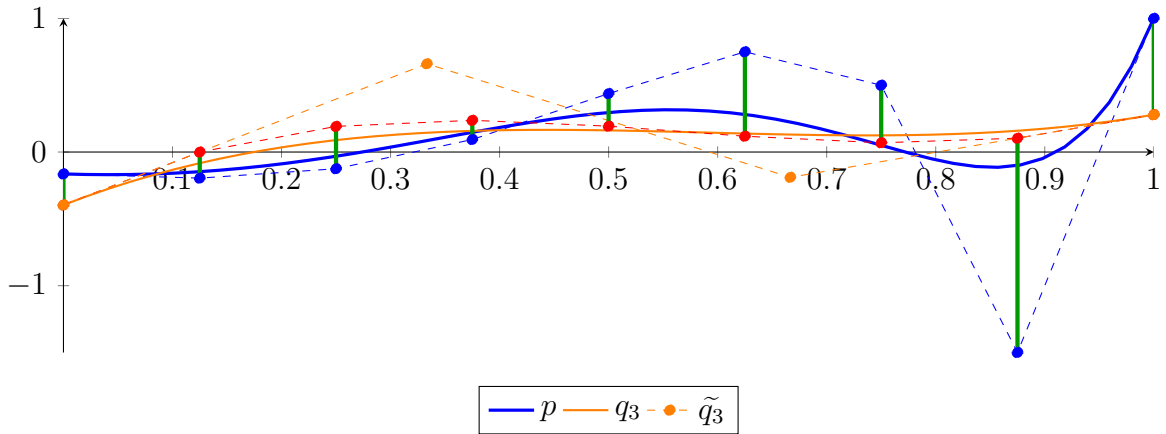
$$= -0.396228B_{0,3} + 0.660275B_{1,3} - 0.189157B_{2,3} + 0.279277B_{3,3}$$

$$\tilde{q}_3 = -8.39477 \cdot 10^{-13}X^8 + 3.46869 \cdot 10^{-12}X^7 - 5.87131 \cdot 10^{-12}X^6 + 5.23145 \cdot 10^{-12}X^5$$

$$- 2.61972 \cdot 10^{-12}X^4 + 3.2238X^3 - 5.7178X^2 + 3.16951X - 0.396228$$

$$= -0.396228B_{0,8} - 3.94571 \cdot 10^{-05}B_{1,8} + 0.191942B_{2,8} + 0.237284B_{3,8}$$

$$+ 0.193554B_{4,8} + 0.11832B_{5,8} + 0.0691513B_{6,8} + 0.103614B_{7,8} + 0.279277B_{8,8}$$



The maximum difference of the Bézier coefficients is  $\delta = 1.60361$ .

**Bounding polynomials  $M$  and  $m$ :**

$$M = 3.2238X^3 - 5.7178X^2 + 3.16951X + 1.20739$$

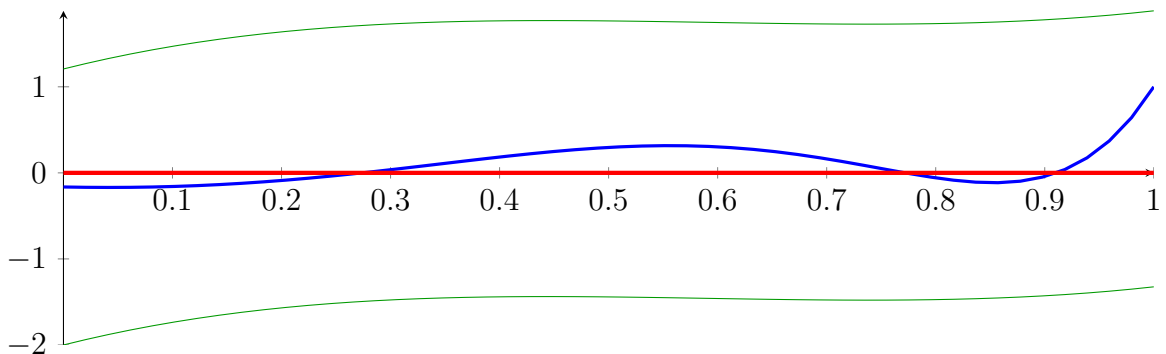
$$m = 3.2238X^3 - 5.7178X^2 + 3.16951X - 1.99984$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{-0.251095\}$$

$$N(m) = \{1.38725\}$$

**Intersection intervals:**



[0, 1]

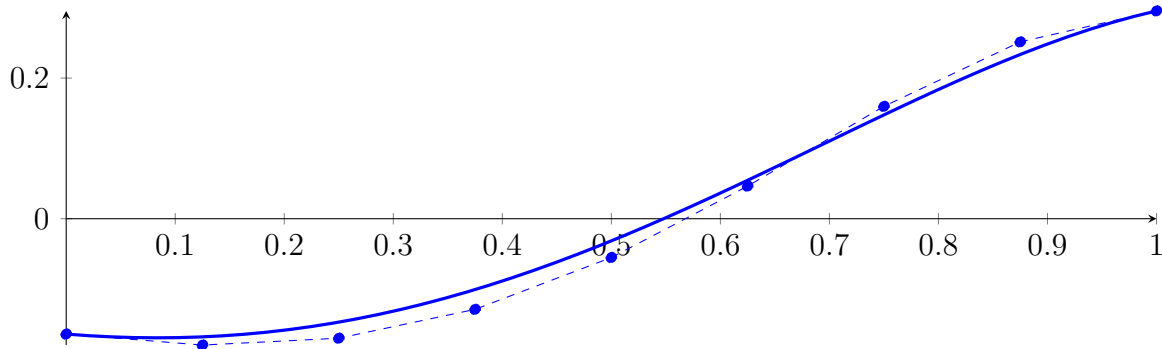
Longest intersection interval: 1

⇒ Bisection: first half [0.5, 0.75] und second half [0.75, 1]

### 3.12 Recursion Branch 1 2 1 on the First Half [0.5, 0.75]

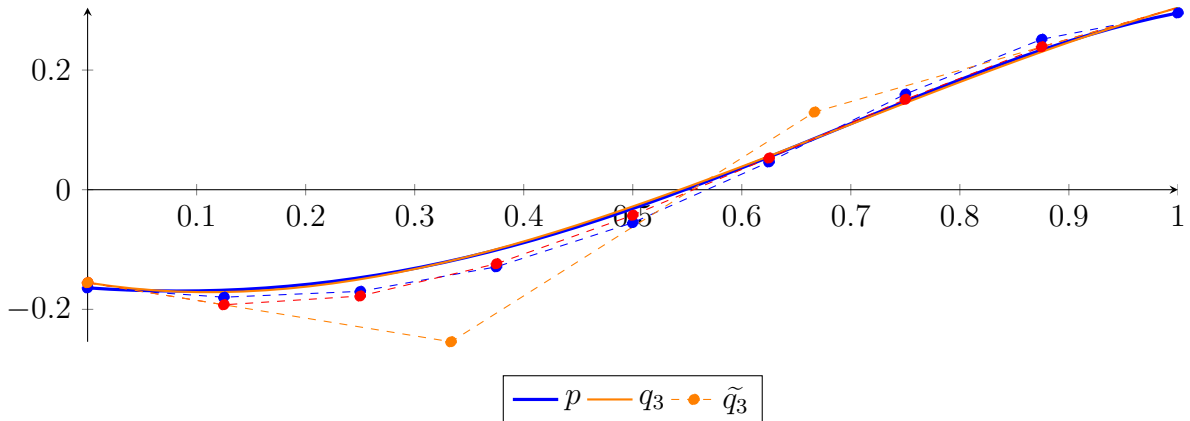
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 0.0323181X^8 + 0.00878906X^7 - 0.0786133X^6 - 0.109375X^5 \\
 &\quad - 0.307617X^4 + 0.328125X^3 + 0.710937X^2 - 0.125X - 0.164063 \\
 &= -0.164063B_{0,8}(X) - 0.179688B_{1,8}(X) - 0.169922B_{2,8}(X) - 0.128906B_{3,8}(X) - 0.0551758B_{4,8}(X) \\
 &\quad + 0.0463867B_{5,8}(X) + 0.15979B_{6,8}(X) + 0.251465B_{7,8}(X) + 0.295502B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -0.69238X^3 + 1.44888X^2 - 0.296882X - 0.15527 \\
 &= -0.15527B_{0,3} - 0.254231B_{1,3} + 0.129769B_{2,3} + 0.304349B_{3,3} \\
 \tilde{q}_3 &= 3.41283 \cdot 10^{-13}X^8 - 1.37986 \cdot 10^{-12}X^7 + 2.28101 \cdot 10^{-12}X^6 - 1.97899 \cdot 10^{-12}X^5 \\
 &\quad + 9.60973 \cdot 10^{-13}X^4 - 0.69238X^3 + 1.44888X^2 - 0.296882X - 0.15527 \\
 &= -0.15527B_{0,8} - 0.19238B_{1,8} - 0.177745B_{2,8} - 0.123728B_{3,8} - 0.0426922B_{4,8} \\
 &\quad + 0.052997B_{5,8} + 0.150976B_{6,8} + 0.238882B_{7,8} + 0.304349B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.0126928$ .

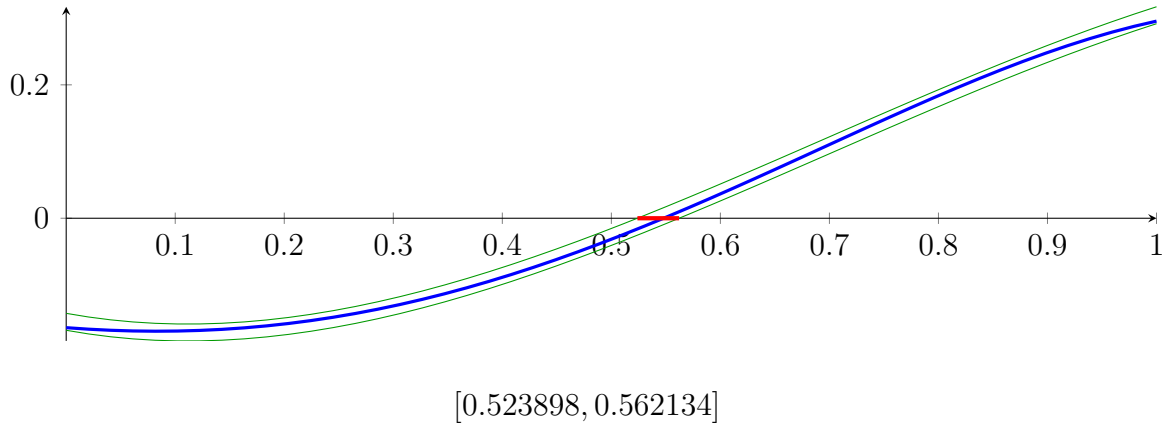
Bounding polynomials  $M$  and  $m$ :

$$\begin{aligned}
 M &= -0.69238X^3 + 1.44888X^2 - 0.296882X - 0.142577 \\
 m &= -0.69238X^3 + 1.44888X^2 - 0.296882X - 0.167963
 \end{aligned}$$

Root of  $M$  and  $m$ :

$$N(M) = \{-0.219773, 0.523898, 1.78848\} \quad N(m) = \{-0.243294, 0.562134, 1.77377\}$$

Intersection intervals:



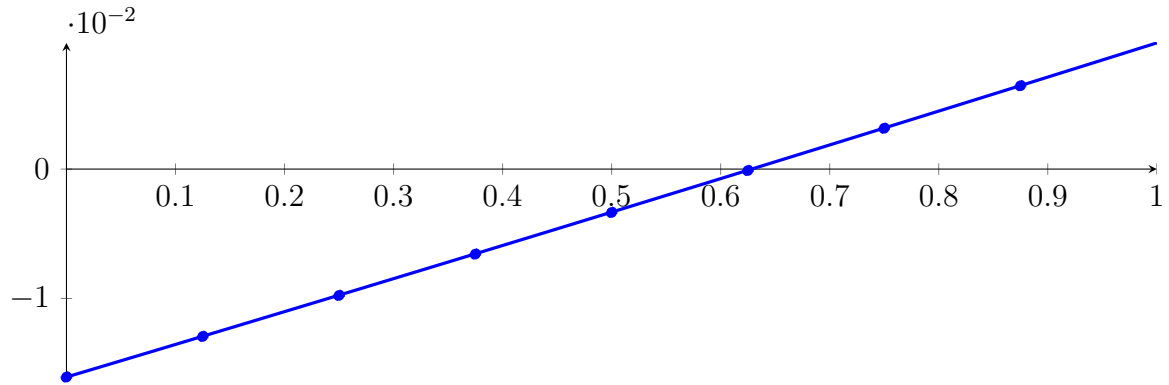
Longest intersection interval: 0.0382362

⇒ Selective recursion: interval 1: [0.630974, 0.640534],

### 3.13 Recursion Branch 1 2 1 1 in Interval 1: [0.630974, 0.640534]

Normalized monomial und Bézier representations and the Bézier polygon:

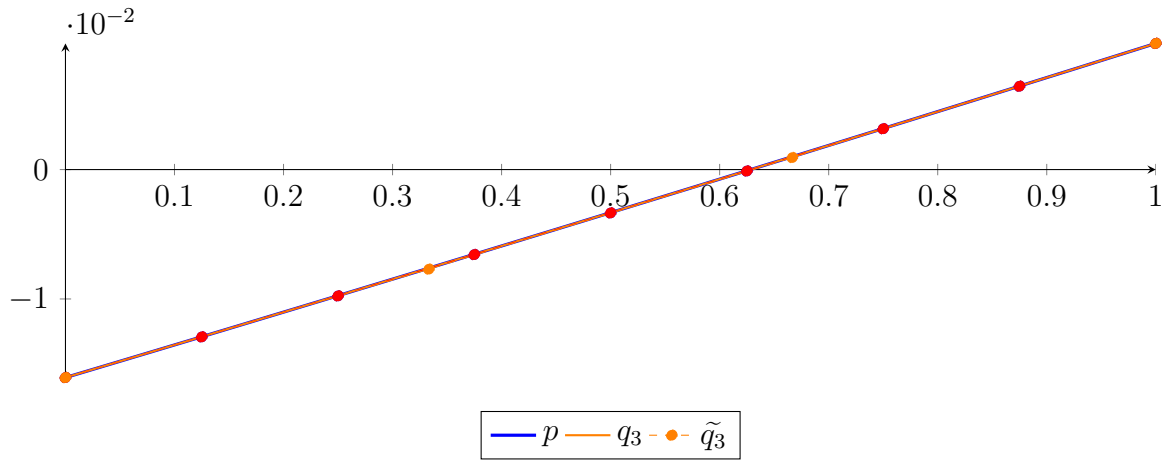
$$\begin{aligned}
p &= 1.47654 \cdot 10^{-13} X^8 + 1.72349 \cdot 10^{-11} X^7 + 6.31209 \cdot 10^{-10} X^6 - 3.72586 \cdot 10^{-09} X^5 - 1.50289 \\
&\quad \cdot 10^{-06} X^4 - 4.18253 \cdot 10^{-05} X^3 + 0.000730929 X^2 + 0.0251374 X - 0.0160745 \\
&= -0.0160745 B_{0,8}(X) - 0.0129323 B_{1,8}(X) - 0.00976402 B_{2,8}(X) \\
&\quad - 0.00657039 B_{3,8}(X) - 0.00335216 B_{4,8}(X) - 0.000110142 B_{5,8}(X) \\
&\quad + 0.00315487 B_{6,8}(X) + 0.00644203 B_{7,8}(X) + 0.0097505 B_{8,8}(X)
\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
q_3 &= -4.48393 \cdot 10^{-05} X^3 + 0.000732868 X^2 + 0.0251369 X - 0.0160744 \\
&= -0.0160744 B_{0,3} - 0.00769546 B_{1,3} + 0.000927802 B_{2,3} + 0.00975052 B_{3,3}
\end{aligned}$$

$$\begin{aligned}
\tilde{q}_3 &= 6.78854 \cdot 10^{-15} X^8 - 2.69897 \cdot 10^{-14} X^7 + 4.37265 \cdot 10^{-14} X^6 - 3.68847 \cdot 10^{-14} X^5 \\
&\quad + 1.70901 \cdot 10^{-14} X^4 - 4.48393 \cdot 10^{-05} X^3 + 0.000732868 X^2 + 0.0251369 X - 0.0160744 \\
&= -0.0160744 B_{0,8} - 0.0129323 B_{1,8} - 0.00976403 B_{2,8} - 0.00657037 B_{3,8} - 0.00335214 B_{4,8} \\
&\quad - 0.000110128 B_{5,8} + 0.00315485 B_{6,8} + 0.006442 B_{7,8} + 0.00975052 B_{8,8}
\end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 3.23628 \cdot 10^{-08}$ .

**Bounding polynomials  $M$  and  $m$ :**

$$M = -4.48393 \cdot 10^{-05} X^3 + 0.000732868 X^2 + 0.0251369 X - 0.0160744$$

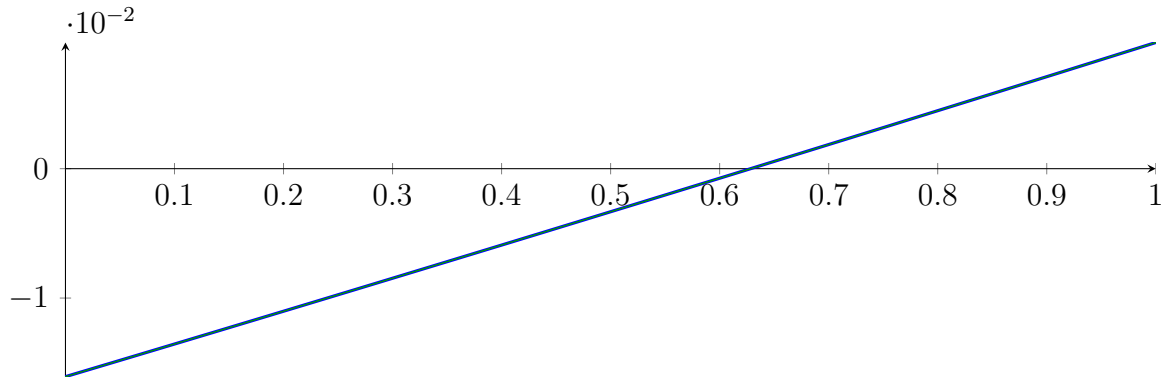
$$m = -4.48393 \cdot 10^{-05} X^3 + 0.000732868 X^2 + 0.0251369 X - 0.0160745$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{-17.2861, 0.628403, 33.002\}$$

$$N(m) = \{-17.2861, 0.628406, 33.002\}$$

**Intersection intervals:**



$$[0.628403, 0.628406]$$

Longest intersection interval:  $2.48897 \cdot 10^{-06}$

$\implies$  Selective recursion: interval 1:  $[0.636981, 0.636981]$ ,

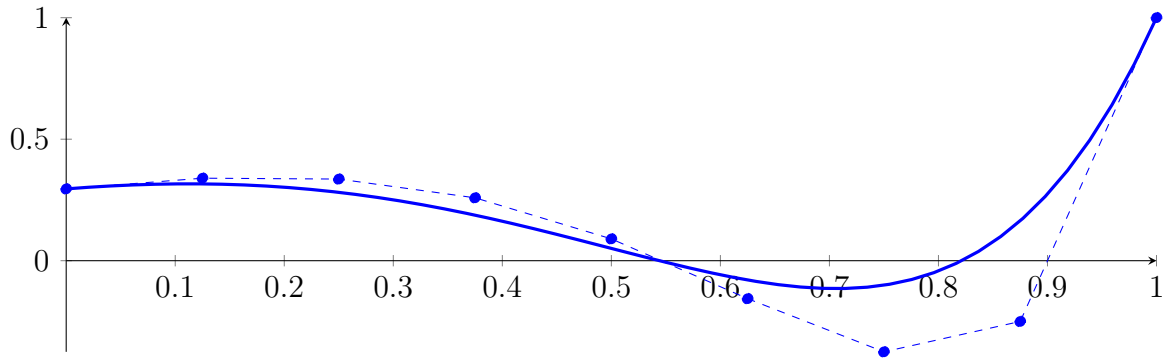
### 3.14 Recursion Branch 1 2 1 1 1 in Interval 1: $[0.636981, 0.636981]$

Found root in interval  $[0.636981, 0.636981]$  at recursion depth 5!

### 3.15 Recursion Branch 1 2 2 on the Second Half $[0.75, 1]$

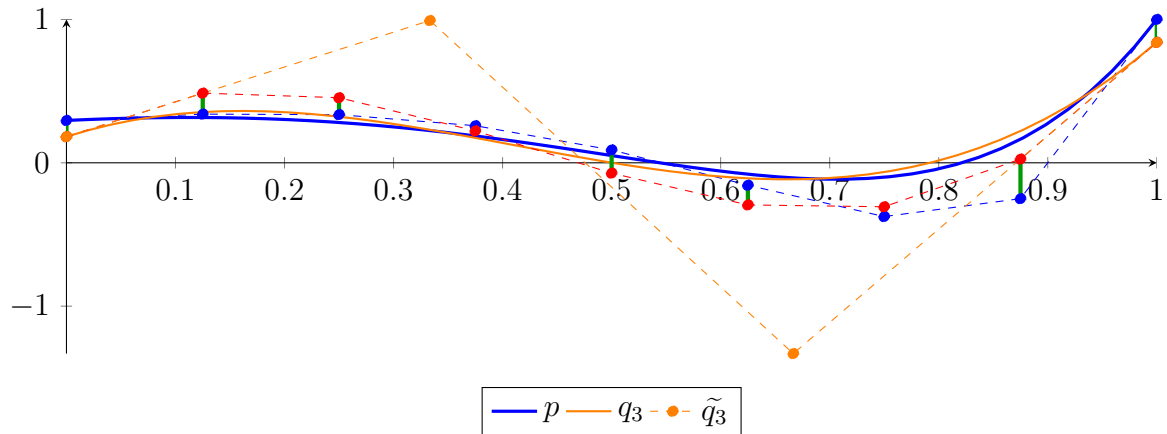
**Normalized monomial und Bézier representations and the Bézier polygon:**

$$\begin{aligned} p &= 0.0323181X^8 + 0.267334X^7 + 0.887817X^6 + 1.41333X^5 \\ &\quad + 0.536194X^4 - 1.45093X^3 - 1.33386X^2 + 0.352295X + 0.295502 \\ &= 0.295502B_{0,8}(X) + 0.339539B_{1,8}(X) + 0.335937B_{2,8}(X) + 0.258789B_{3,8}(X) \\ &\quad + 0.0898437B_{4,8}(X) - 0.15625B_{5,8}(X) - 0.375B_{6,8}(X) - 0.25B_{7,8}(X) + 1B_{8,8}(X) \end{aligned}$$



**Degree reduction and raising:**

$$\begin{aligned}
 q_3 &= 7.62711X^3 - 9.39986X^2 + 2.43046X + 0.18221 \\
 &= 0.18221B_{0,3} + 0.992362B_{1,3} - 1.33077B_{2,3} + 0.839916B_{3,3} \\
 \tilde{q}_3 &= 5.87539 \cdot 10^{-13}X^8 - 2.08028 \cdot 10^{-12}X^7 + 2.89378 \cdot 10^{-12}X^6 - 1.98651 \cdot 10^{-12}X^5 \\
 &\quad + 6.86876 \cdot 10^{-13}X^4 + 7.62711X^3 - 9.39986X^2 + 2.43046X + 0.18221 \\
 &= 0.18221B_{0,8} + 0.486017B_{1,8} + 0.454115B_{2,8} + 0.222702B_{3,8} - 0.072024B_{4,8} \\
 &\quad - 0.293864B_{5,8} - 0.306619B_{6,8} + 0.0259079B_{7,8} + 0.839916B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.275908$ .

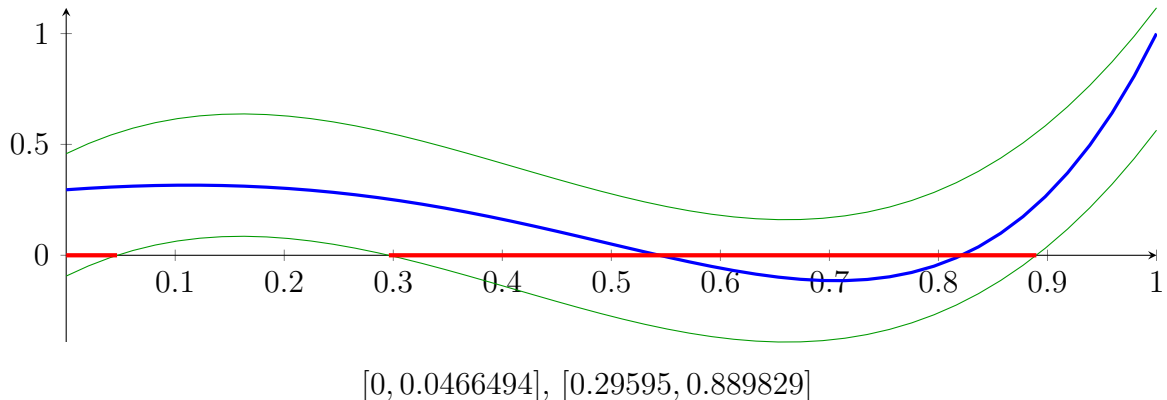
**Bounding polynomials  $M$  and  $m$ :**

$$\begin{aligned}
 M &= 7.62711X^3 - 9.39986X^2 + 2.43046X + 0.458118 \\
 m &= 7.62711X^3 - 9.39986X^2 + 2.43046X - 0.0936979
 \end{aligned}$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{-0.123543\} \qquad N(m) = \{0.0466494, 0.29595, 0.889829\}$$

**Intersection intervals:**



Longest intersection interval: 0.593879

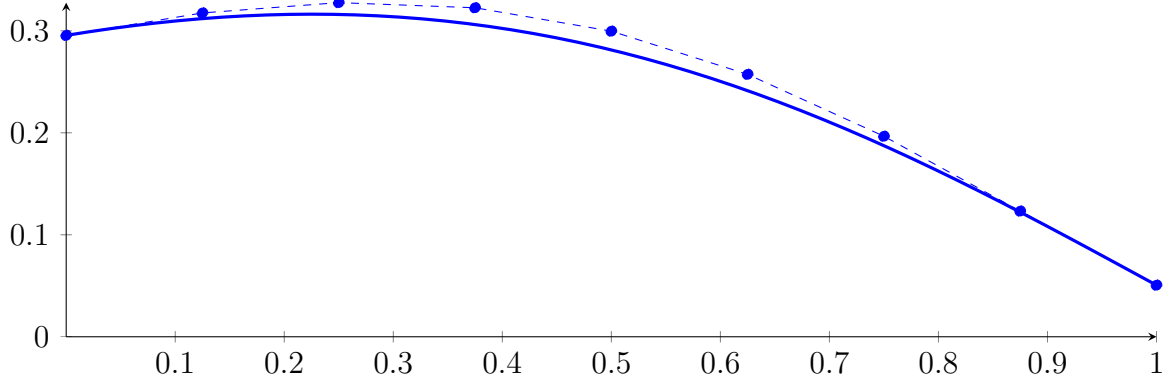
$\implies$  Bisection: first half [0.75, 0.875] und second half [0.875, 1]



### 3.16 Recursion Branch 1 2 2 1 on the First Half [0.75, 0.875]

Normalized monomial und Bézier representations and the Bézier polygon:

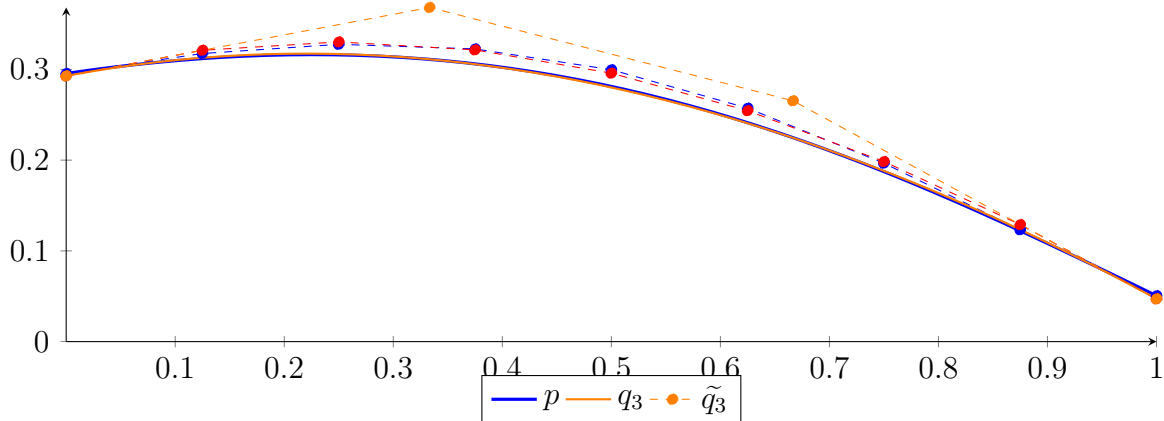
$$\begin{aligned}
 p &= 0.000126243X^8 + 0.00208855X^7 + 0.0138721X^6 + 0.0441666X^5 \\
 &\quad + 0.0335121X^4 - 0.181366X^3 - 0.333466X^2 + 0.176147X + 0.295502 \\
 &= 0.295502B_{0,8}(X) + 0.31752B_{1,8}(X) + 0.327629B_{2,8}(X) + 0.32259B_{3,8}(X) + 0.299643B_{4,8}(X) \\
 &\quad + 0.257295B_{5,8}(X) + 0.196605B_{6,8}(X) + 0.123225B_{7,8}(X) + 0.0505832B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= 0.0628365X^3 - 0.53481X^2 + 0.226271X + 0.292825 \\
 &= 0.292825B_{0,3} + 0.368248B_{1,3} + 0.265402B_{2,3} + 0.0471216B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= -6.09365 \cdot 10^{-13}X^8 + 2.44448 \cdot 10^{-12}X^7 - 4.00243 \cdot 10^{-12}X^6 + 3.42872 \cdot 10^{-12}X^5 \\
 &\quad - 1.6319 \cdot 10^{-12}X^4 + 0.0628365X^3 - 0.53481X^2 + 0.226271X + 0.292825 \\
 &= 0.292825B_{0,8} + 0.321109B_{1,8} + 0.330292B_{2,8} + 0.321497B_{3,8} + 0.295846B_{4,8} \\
 &\quad + 0.254461B_{5,8} + 0.198464B_{6,8} + 0.128977B_{7,8} + 0.0471216B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.00575122$ .

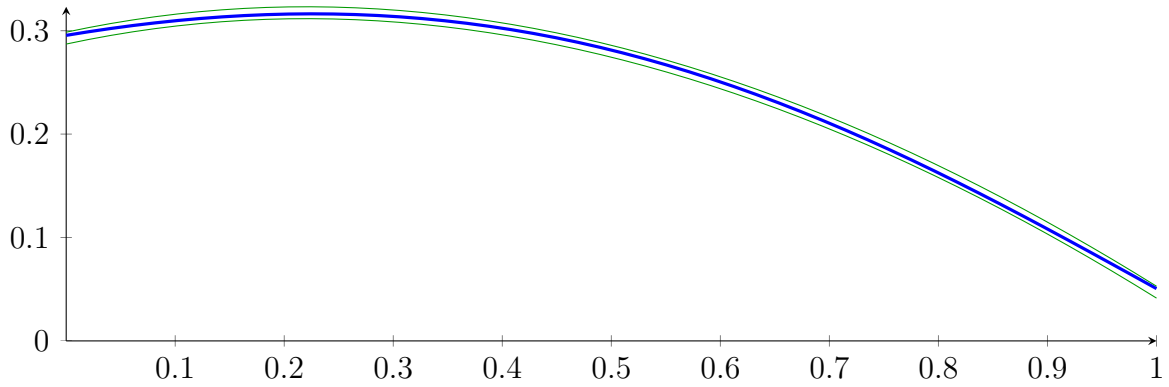
Bounding polynomials  $M$  and  $m$ :

$$\begin{aligned}
 M &= 0.0628365X^3 - 0.53481X^2 + 0.226271X + 0.298576 \\
 m &= 0.0628365X^3 - 0.53481X^2 + 0.226271X + 0.287074
 \end{aligned}$$

Root of  $M$  and  $m$ :

$$N(M) = \{-0.552168, 1.0776, 7.98571\} \quad N(m) = \{-0.538884, 1.06122, 7.98881\}$$

Intersection intervals:

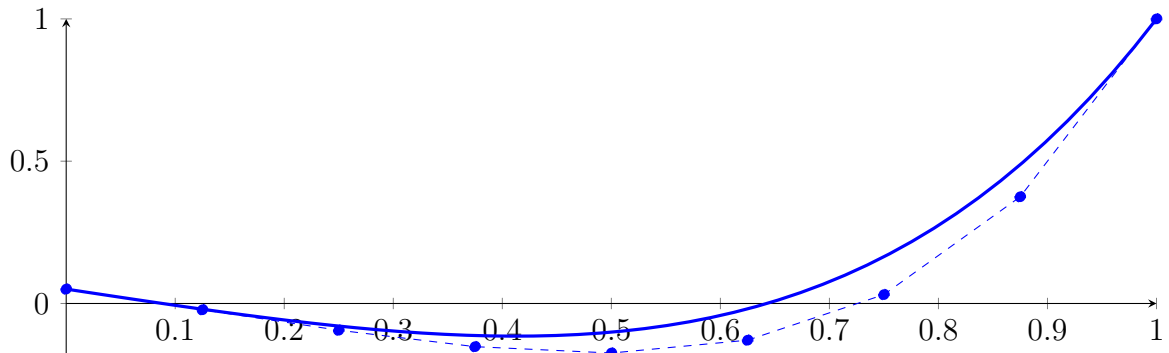


No intersection intervals with the  $x$  axis.

### 3.17 Recursion Branch 1 2 2 2 on the Second Half $[0.875, 1]$

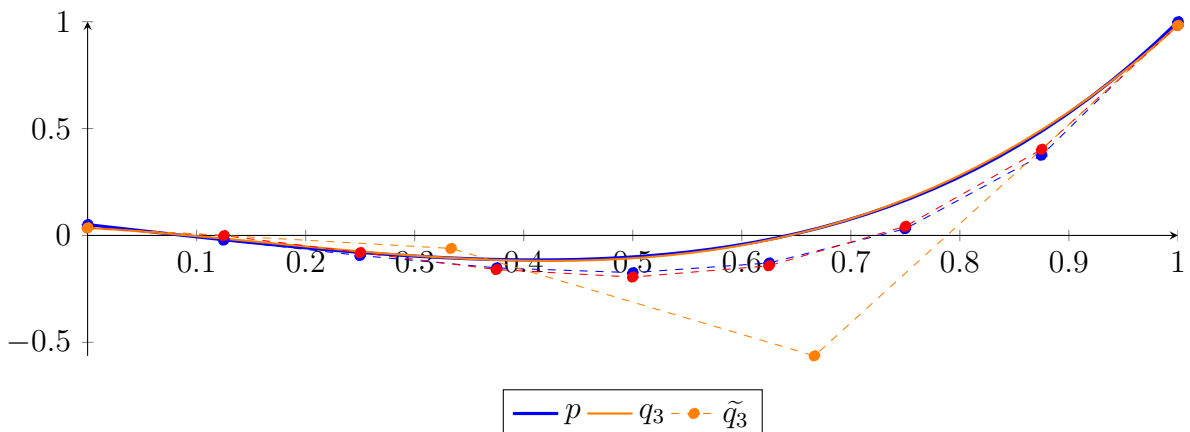
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 0.000126243X^8 + 0.00309849X^7 + 0.0320268X^6 + 0.178329X^5 \\
 &\quad + 0.544363X^4 + 0.75196X^3 + 0.0206513X^2 - 0.581138X + 0.0505832 \\
 &= 0.0505832B_{0,8}(X) - 0.022059B_{1,8}(X) - 0.0939636B_{2,8}(X) - 0.151703B_{3,8}(X) \\
 &\quad - 0.174072B_{4,8}(X) - 0.128906B_{5,8}(X) + 0.03125B_{6,8}(X) + 0.375B_{7,8}(X) + 1B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= 2.4548X^3 - 1.21915X^2 - 0.288567X + 0.0354236 \\
 &= 0.0354236B_{0,3} - 0.0607652B_{1,3} - 0.563336B_{2,3} + 0.982513B_{3,3} \\
 \tilde{q}_3 &= 1.34022 \cdot 10^{-12}X^8 - 5.25494 \cdot 10^{-12}X^7 + 8.38443 \cdot 10^{-12}X^6 - 6.9863 \cdot 10^{-12}X^5 \\
 &\quad + 3.24323 \cdot 10^{-12}X^4 + 2.4548X^3 - 1.21915X^2 - 0.288567X + 0.0354236 \\
 &= 0.0354236B_{0,8} - 0.000647215B_{1,8} - 0.0802589B_{2,8} - 0.159576B_{3,8} \\
 &\quad - 0.194762B_{4,8} - 0.141982B_{5,8} + 0.0425998B_{6,8} + 0.40282B_{7,8} + 0.982513B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is  $\delta = 0.0278195$ .

**Bounding polynomials  $M$  and  $m$ :**

$$M = 2.4548X^3 - 1.21915X^2 - 0.288567X + 0.0632431$$

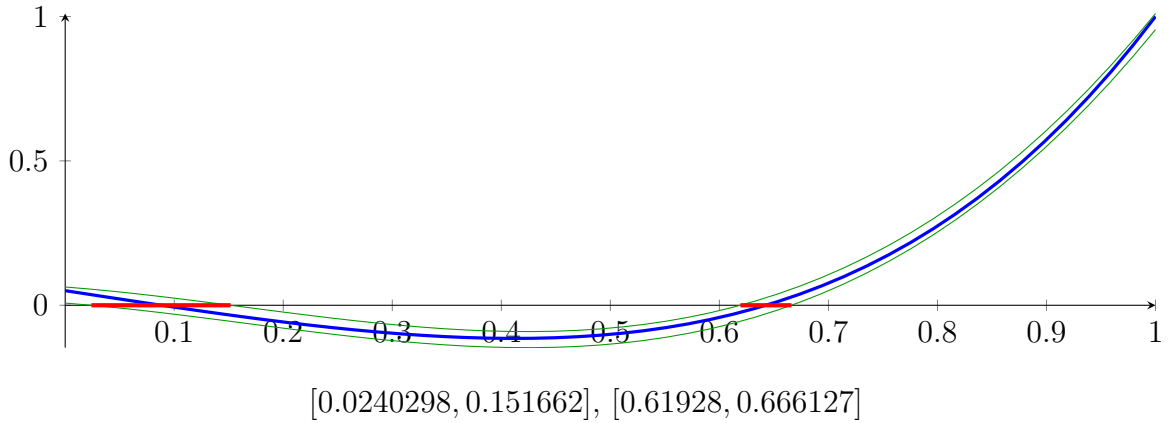
$$m = 2.4548X^3 - 1.21915X^2 - 0.288567X + 0.00760409$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{-0.274305, 0.151662, 0.61928\}$$

$$N(m) = \{-0.19352, 0.0240298, 0.666127\}$$

**Intersection intervals:**



Longest intersection interval: 0.127632

$\implies$  Selective recursion: interval 1: [0.878004, 0.893958], interval 2: [0.95241, 0.958266],

### 3.18 Recursion Branch 1 2 2 1 in Interval 1: [0.878004, 0.893958]

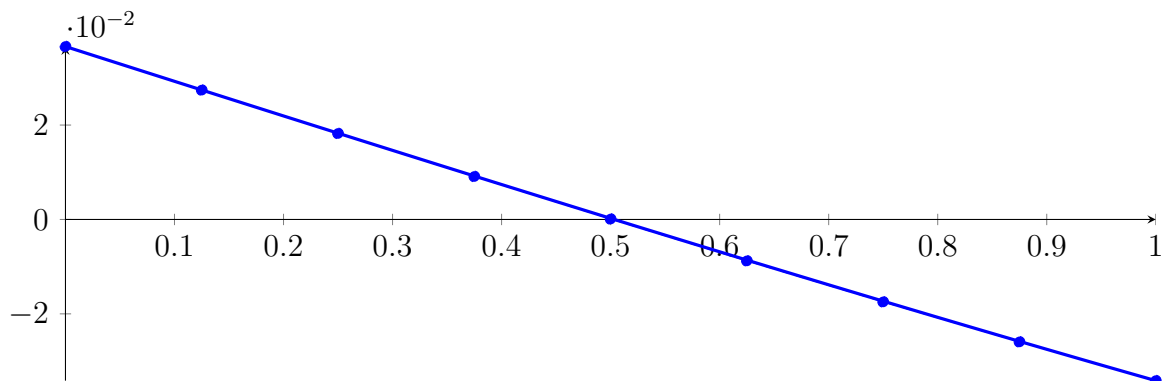
**Normalized monomial und Bézier representations and the Bézier polygon:**

$$p = 8.88968 \cdot 10^{-12} X^8 + 1.72289 \cdot 10^{-09} X^7 + 1.40705 \cdot 10^{-07} X^6 + 6.19744 \cdot 10^{-06} X^5$$

$$+ 0.000150213 X^4 + 0.00167436 X^3 + 0.00125059 X^2 - 0.073875 X + 0.0366412$$

$$= 0.0366412 B_{0,8}(X) + 0.0274068 B_{1,8}(X) + 0.0182171 B_{2,8}(X) + 0.00910195 B_{3,8}(X) + 9.34104$$

$$\cdot 10^{-05} B_{4,8}(X) - 0.00877422 B_{5,8}(X) - 0.0174643 B_{6,8}(X) - 0.0259376 B_{7,8}(X) - 0.0341523 B_{8,8}(X)$$



**Degree reduction and raising:**

$$q_3 = 0.00199248 X^3 + 0.00104224 X^2 - 0.0738283 X + 0.0366388$$

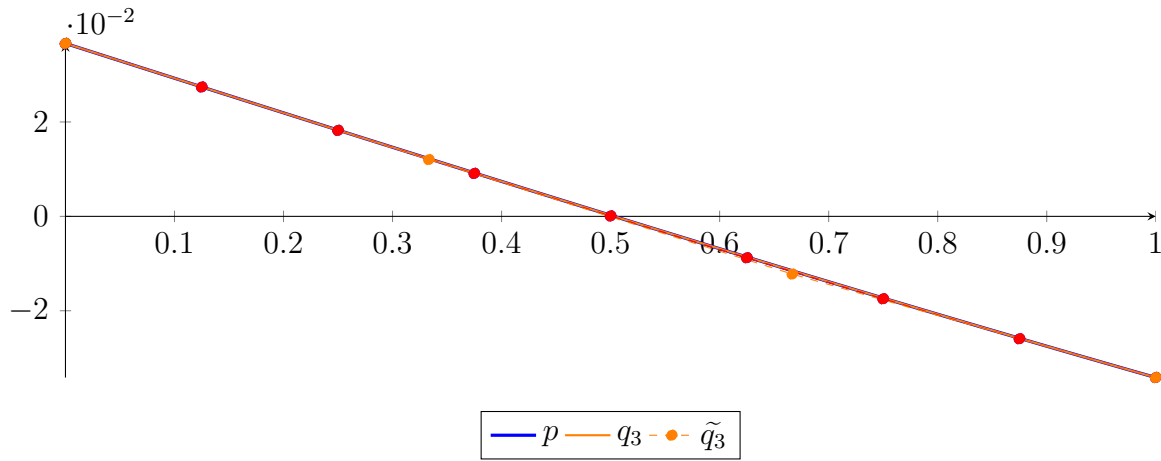
$$= 0.0366388 B_{0,3} + 0.0120294 B_{1,3} - 0.0122326 B_{2,3} - 0.0341547 B_{3,3}$$

$$\tilde{q}_3 = -4.51505 \cdot 10^{-15} X^8 + 1.73535 \cdot 10^{-14} X^7 - 2.69943 \cdot 10^{-14} X^6 + 2.14824 \cdot 10^{-14} X^5$$

$$- 9.00982 \cdot 10^{-15} X^4 + 0.00199248 X^3 + 0.00104224 X^2 - 0.0738283 X + 0.0366388$$

$$= 0.0366388 B_{0,8} + 0.0274103 B_{1,8} + 0.018219 B_{2,8} + 0.00910048 B_{3,8} + 9.03569$$

$$\cdot 10^{-05} B_{4,8} - 0.00877581 B_{5,8} - 0.0174624 B_{6,8} - 0.0259339 B_{7,8} - 0.0341547 B_{8,8}$$



The maximum difference of the Bézier coefficients is  $\delta = 3.63643 \cdot 10^{-06}$ .

**Bounding polynomials  $M$  and  $m$ :**

$$M = 0.00199248X^3 + 0.00104224X^2 - 0.0738283X + 0.0366425$$

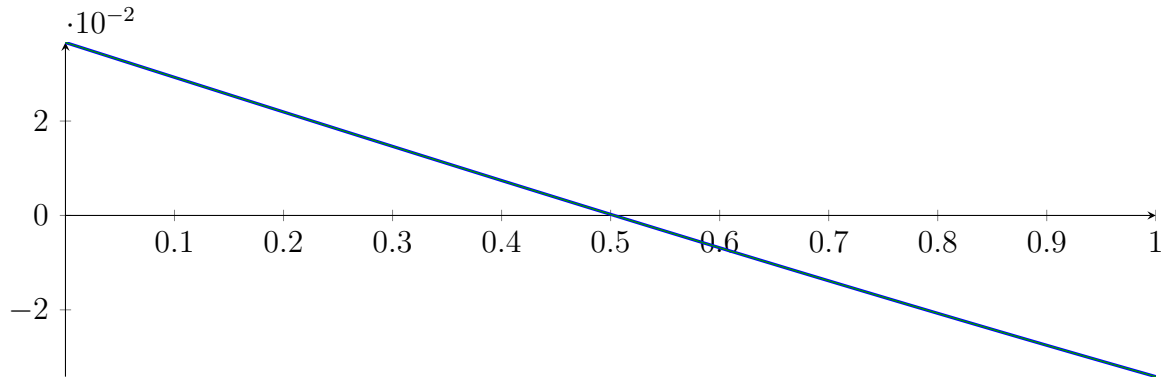
$$m = 0.00199248X^3 + 0.00104224X^2 - 0.0738283X + 0.0366352$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{-6.57953, 0.503338, 5.5531\}$$

$$N(m) = \{-6.57949, 0.503236, 5.55316\}$$

**Intersection intervals:**



$$[0.503236, 0.503338]$$

Longest intersection interval: 0.000102054

$\implies$  Selective recursion: [interval 1: \[0.886032, 0.886034\]](#),

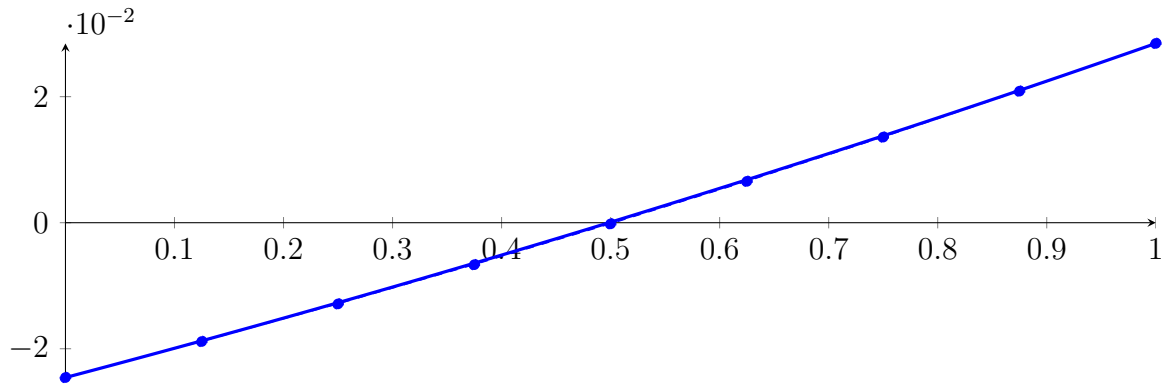
### 3.19 Recursion Branch 1 2 2 2 1 1 in Interval 1: [0.886032, 0.886034]

Found root in interval [0.886032, 0.886034] at recursion depth 6!

### 3.20 Recursion Branch 1 2 2 2 2 in Interval 2: [0.95241, 0.958266]

**Normalized monomial und Bézier representations and the Bézier polygon:**

$$\begin{aligned} p &= 2.92844 \cdot 10^{-15} X^8 + 1.844 \cdot 10^{-12} X^7 + 4.94833 \cdot 10^{-10} X^6 + 7.30964 \cdot 10^{-08} X^5 + 6.29902 \\ &\quad \cdot 10^{-06} X^4 + 0.000303605 X^3 + 0.00695825 X^2 + 0.0457521 X - 0.0245702 \\ &= -0.0245702 B_{0,8}(X) - 0.0188512 B_{1,8}(X) - 0.0128837 B_{2,8}(X) \\ &\quad - 0.00666221 B_{3,8}(X) - 0.000181311 B_{4,8}(X) + 0.00656463 B_{5,8}(X) \\ &\quad + 0.0135813 B_{6,8}(X) + 0.0208745 B_{7,8}(X) + 0.0284501 B_{8,8}(X) \end{aligned}$$



**Degree reduction and raising:**

$$q_3 = 0.000316408X^3 + 0.00694998X^2 + 0.045754X - 0.0245703$$

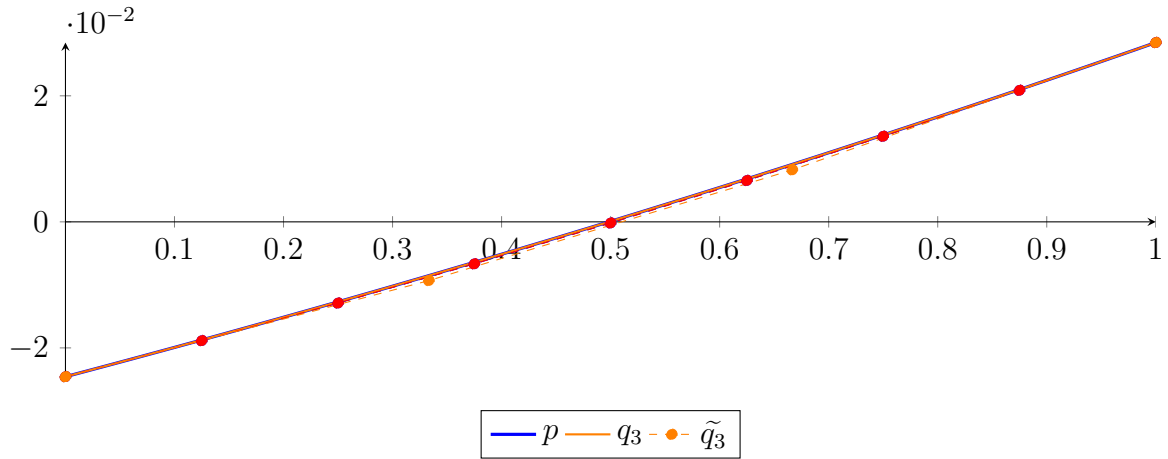
$$= -0.0245703B_{0,3} - 0.00931897B_{1,3} + 0.00824901B_{2,3} + 0.0284501B_{3,3}$$

$$\tilde{q}_3 = 8.43466 \cdot 10^{-15}X^8 - 3.30538 \cdot 10^{-14}X^7 + 5.26942 \cdot 10^{-14}X^6 - 4.3669 \cdot 10^{-14}X^5$$

$$+ 1.98998 \cdot 10^{-14}X^4 + 0.000316408X^3 + 0.00694998X^2 + 0.045754X - 0.0245703$$

$$= -0.0245703B_{0,8} - 0.018851B_{1,8} - 0.0128836B_{2,8} - 0.00666227B_{3,8} - 0.00018143B_{4,8}$$

$$+ 0.00656457B_{5,8} + 0.0135814B_{6,8} + 0.0208747B_{7,8} + 0.0284501B_{8,8}$$



The maximum difference of the Bézier coefficients is  $\delta = 1.39755 \cdot 10^{-07}$ .

**Bounding polynomials  $M$  and  $m$ :**

$$M = 0.000316408X^3 + 0.00694998X^2 + 0.045754X - 0.0245702$$

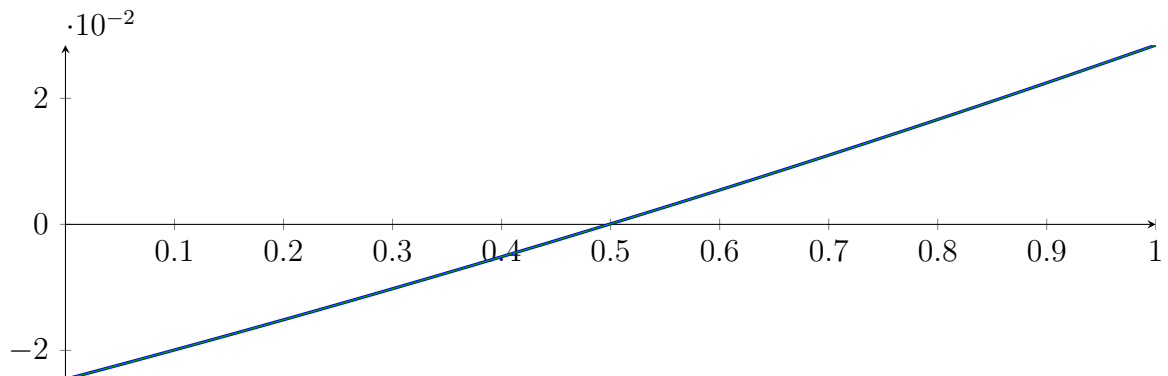
$$m = 0.000316408X^3 + 0.00694998X^2 + 0.045754X - 0.0245704$$

**Root of  $M$  and  $m$ :**

$$N(M) = \{0.498415\}$$

$$N(m) = \{0.498421\}$$

**Intersection intervals:**



[0.498415, 0.498421]

Longest intersection interval:  $5.28196 \cdot 10^{-06}$

$\implies$  Selective recursion: interval 1: [0.955329, 0.955329],

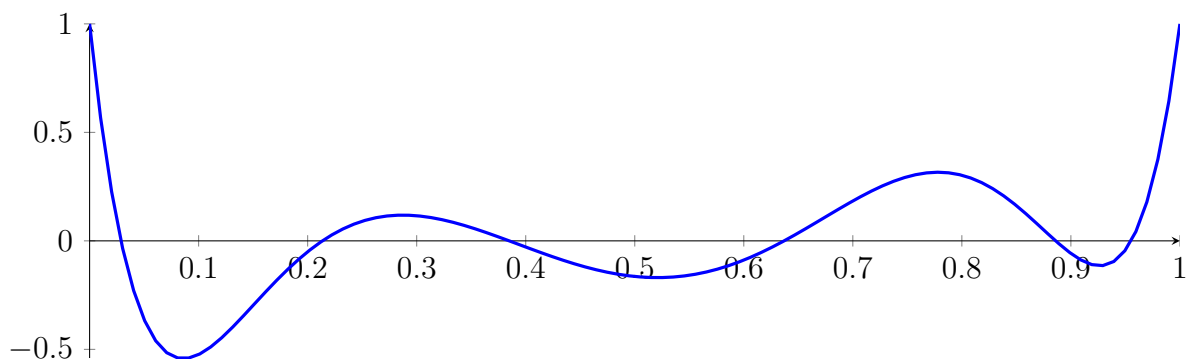
### **3.21 Recursion Branch 1 2 2 2 2 1 in Interval 1: [0.955329, 0.955329]**

Found root in interval [0.955329, 0.955329] at recursion depth 6!

### 3.22 Result: 6 Root Intervals

Input Polynomial on Interval  $[0, 1]$

$$p = 2118X^8 - 8328X^7 + 14000X^6 - 13216X^5 + 7630X^4 - 2688X^3 + 532X^2 - 48X + 1$$



**Result: Root Intervals**

$$[0.0287822, 0.0287874], [0.213227, 0.213918], [0.384836, 0.384836], [0.636981, 0.636981], \\ [0.886032, 0.886034], [0.955329, 0.955329]$$

with precision  $\varepsilon = 0.001$ .